U. S. DEPARTMENT OF COMMERCE
COAST AND GEODETIC SURVEY

MANUAL OF HARMONIC ANALYSIS AND PREDICTION OF TIDES

SPECIAL PUBLICATION No. 98
REVISED (1940) EDITION

U.S. DEPARTMENT OF COMMERCE
JESSE H. JONES, Secretary

COAST AND GEODETIC SURVEY
LEO OTIS COLBERT, Director

Special Publication No. 98
Revised (1940) Edition
(Reprinted 1958 with corrections)

MANUAL OF HARMONIC ANALYSIS AND PREDICTION OF TIDES

BY
PAUL SCHUREMAN
Senior Mathematician

Reprinted June 2001

UNITED STATES
GOVERNMENT PRINTING OFFICE
WASHINGTON : 1958

PREFACE

This volume was designed primarily as a working manual for use in the United States Coast and Geodetic Survey and describes the procedure used in this office for the harmonic analysis and prediction of tides and tidal currents. It is based largely upon the works of Sir William Thomson, Prof. George H. Darwin, and Dr. Rollin A. Harris. In recent years there also has been considerable work done on this subject by Dr. A. T. Doodson, of the Tidal Institute of the University of Liverpool.

The first edition of the present work was published in 1924. In this revised edition there has been a rearrangement of the material in the first part of the volume to bring out more clearly the development of the tidal forces. Tables of astronomical data and other tables to facilitate the computations have been retained with a few revisions and additions and there has been added a list of symbols used in the work.

The collection of tidal harmonic constants for the world that appeared in the earlier edition has been omitted altogether because the work of maintaining such a list has now been taken over by the International Hydrographic Bureau at Monaco. These constants are now published in International Hydrographic Bureau Special Publication No. 26, which consists of a collection of loose sheets which permit the addition of new constants as they become available.

Special acknowledgment is due Walter B. Zerbe, associate mathematician of the Division of Tides and Currents, who reviewed the manuscript of this edition and offered many valuable suggestions.

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MANUAL OF HARMONIC ANALYSIS AND PREDICTION OF TIDES

INTRODUCTION

HISTORICAL STATEMENT

1. Sir William Thomson (Lord Kelvin) devised the method of reduction of tides by harmonic analysis about the year 1867. The principle upon which the system is based—which is that any periodic motion or oscillation can always be resolved into the sum of a series of simple harmonic motions—is said to have been discovered by Eudoxas as early as 356 B. C., when he explained the apparently irregular motions of the planets by combinations of uniform circular motions.[1] In the early part of the nineteenth century Laplace recognized the existence of partial tides that might be expressed by the cosine of an angle increasing uniformly with the time, and also applied the essential principles of the harmonic analysis to the reduction of high and low waters. Dr. Thomas Young suggested the importance of observing and analyzing the entire tidal curve rather than the high and low waters only. Sir George B. Airy also had an important part in laying the foundation for the harmonic analysis of the tides. To Sir William Thomson, however, we may give the credit for having placed the analysis on a practical basis.

2. In 1867 the British Association for the Advancement of Science appointed a committee for the purpose of promoting the extension, improvement, and harmonic analysis of tidal observations. The report on the subject was prepared by Sir William Thomson and was published in the Report of the British Association for the Advancement of Science in 1868. Supplementary reports were made from time to time by the tidal committee and published in subsequent reports of the British association. A few years later a committe, consisting of Profs. G. H. Darwin and J. C. Adams, drew up a very full report on the subject, which was published in the Report of the British Association for the Advancement of Science in 1883.

3. Among the American mathematicians who have had an important part in the development of this subject may be named Prof. William Ferrel and Dr. Rollin A. Harris, both of whom were associated with the U. S. Coast and Geodetic Survey. The Tidal Researches, by Professor Ferrel, was published in 1874, and additional articles on the harmonic analysis by the same author appeared from time to time in the annual reports of the Superintendent of the Coast and Geodetic Survey. The best known work of Doctor Harris is his Manual of Tides, which was published in several parts as appendices to the annual reports of the Superintendent of the Coast and Geodetic Survey. The subject of the harmonic analysis was treated principally in Part II of the Manual which appeared in 1897.

[1] Nautical Science, p. 279, by Charles Lane Poor.

GENERAL EXPLANATION OF TIDAL MOVEMENT

4. That the tidal movement results from the gravitational attraction of the moon and sun acting upon the rotating earth is now a well-established scientific fact. The movement includes both the vertical rise and fall of the tide and the horizontal flow of the tidal currents. It will be shown later that the tide-producing force due to this attraction, when taken in connection with the attraction between the particles of matter which constitute the earth, can be expressed by mathematical formulas based upon the well-known laws of gravitation.

5. Although the acting forces are well understood, the resultant tidal movement is exceedingly complicated because of the irregular distribution of land and water on the earth and the retarding effects of friction and inertia. Contrary to the popular idea of a progressive tidal wave following the moon around the earth, the basic tidal movement as evidenced by observations at numerous points along the shores of the oceans consists of a number of oscillating areas, the movement being somewhat similar to that in a pan of water that has been tilted. Such oscillations are technically known as stationary waves. The complex nature of the movement can be appreciated when consideration is given to the fact that such stationary waves may overlap or be superimposed upon each other and may be accompanied by a progressive wave movement.

6. Any basin of water has its natural free period of oscillation depending upon its size and depth. The usual formula for the period of oscillation in a rectangular tank of uniform depth is $2L/\sqrt{gd}$, in which L is the length and d the depth of the tank and g is the acceleration of gravity. When a disturbing force is applied periodically at intervals corresponding to the free period of a body of water, it tends to build up an oscillation of much greater magnitude than would be possible with a single application of the force. The major tidal oscillations have periods approximating the half and the whole lunar day.

HARMONIC TREATMENT OF TIDAL DATA

7. The harmonic analysis of tides is based upon an assumption that the rise and fall of the tide in any locality can be expressed mathematically by the sum of a series of harmonic terms having certain relations to astronomical conditions. A simple harmonic function is a quantity that varies as the cosine of an angle that increases uniformly with time. In the equation $y=A \cos at$, y is an harmonic function of the angle at in which a is a constant and t represents time as measured from some initial epoch. The general equation for the height (h) of the tide at any time (t) may be written

$$h=H_0+A \cos (at+\alpha)+B \cos (bt+\beta)+C \cos (ct+\gamma)+ \text{etc.} \quad (1)$$

in which H_0 is the height of the mean water level above the datum used. Other symbols are explained in the following paragraph.

8. Each cosine term in equation (1) is known as a *constituent* or *component* tide. The coefficients A, B, C, etc. are the *amplitudes* of the constituents and are derived from observed tidal data in each locality. The expression in parentheses is a uniformly-varying angle and its value at any time is called its *phase*. Any constituent term has its maximum positive value when the phase of the angle is zero and a maximum negative value when the phase equals 180°, and the

term becomes zero when the phase equals 90° or 270°. The coefficient of t represents the rate of change in the phase and is called the *speed* of the constituent and is usually expressed in degrees per hour. The time required for a constituent to pass through a complete cycle is known as its *period* and may be obtained by dividing 360° by its speed. The periods and corresponding speeds of the constituents are derived from astronomical data and are independent of the locality of the tide station. The symbols α, β, γ, etc. refer to the initial phases of the constituent angles at the time when t equals zero. The initial phases depend upon locality as well as the instant from which the time is reckoned and their values are derived from tidal observations. *Harmonic analysis* as applied to tides is the process by which the observed tidal data at any place are separated into a number of harmonic constituents. The quantities sought are known as *harmonic constants* and consist of the amplitudes and certain phase relations which will be more fully explained later. *Harmonic prediction* is accomplished by reuniting the elementary constituents in accordance with astronomical relations prevailing at the time for which the predictions are being made.

ASTRONOMICAL DATA

9. In tidal work the only celestial bodies that need be considered are the moon and sun. Although every other celestial body whose gravitational influence reaches the earth creates a theoretical tide-producing force, the greater distance or smaller size of such body renders negligible any effect of this force upon the tides of the earth. In deriving mathematical expressions for the tide-producing forces of the moon and sun, the principal factors to be taken into consideration are the rotation of the earth, the revolution of the moon around the earth, the revolution of the earth around the sun, the inclination of the moon's orbit to the earth's equator, and the obliquity of the ecliptic. Numerical values pertaining to these factors will be found in table 1.

10. The earth rotates on its axis once each day. There are, however, several kinds of days—the sidereal day, the solar day, the lunar day, and the constituent day—depending upon the object used as a reference for the rotation. The *sidereal day* is defined by astronomers as the time required for the rotation of the earth with respect to the vernal equinox. Because of the precession of the equinox, this day differs slightly from the time of rotation with respect to a fixed star, the difference being less than the hundredth part of a second. The *solar day* and *lunar day* are respectively the times required for rotation with respect to the sun and moon. Since the motions of the earth and moon in their orbits are not uniform, the solar and lunar days vary a little in length and their average or mean values are taken as standard units of time. A *constituent day* is the time of the rotation of the earth with respect to a fictitious satellite representing one of the periodic elements in the tidal forces. It approximates in length the lunar or solar day and corresponds to the period of a diurnal constituent or twice the period of a semidiurnal constituent.

11. A *calendar day* is a mean solar day commencing at midnight. Such a calendar day is known also as a *civil day* to distinguish it from the *astronomical day* which commences at noon of the same date.

Prior to the year 1925, the astronomical day was in general use by astronomers for the recording of astronomical data, but beginning with the Ephemeris and Nautical Almanac published in 1925 the civil day has been adopted for the calculations. Each day of whatever kind may be divided into 24 equal parts known as hours which are qualified by the name of the kind of day of which they are a part, as *sidereal hour*, *solar hour*, *lunar hour*, or *constituent hour*.

12. The moon revolves around the earth in an elliptical orbit. Although the average eccentricity of this orbit remains approximately constant for long periods of time, there are a number of perturbations in the moon's motion due, primarily, to the attractive force of the sun. Besides the revolution of the line of apsides and the regression of the nodes which take place more or less slowly, the principal inequalities in the moon's motion which affect the tides are the evection and variation. The evection depends upon the alternate increase and decrease of the eccentricity of the moon's orbit, which is always a maximum when the sun is passing the moon's line of apsides, and a minimum when the sun is at right angles to it. The variation inequality is due mainly to the tangential component of the disturbing force. The period of the revolution of the moon around the earth is called a month. The month is designated as sidereal, tropical, anomalistic, nodical, or synodical, according to whether the revolution is relative to a fixed star, the vernal equinox, the perigee, the ascending node, or the sun. The calendar month is a rough approximation to the synodical month.

13. It is customary to refer to the revolution of the earth around the sun, although it may be more accurately stated that they both revolve around their common center of gravity; but if we imagine the earth as fixed, the sun will describe an apparent path around the earth which is the same in size and form as the orbit of the earth around the sun, and the effect upon the tides would be the same. This orbit is an ellipse with an eccentricity that changes so slowly that it may be considered as practically constant. The period of the revolution of the earth around the sun is a year, but there are several kinds of years. The *sidereal year* is a revolution with respect to a fixed star, the *tropical year* is a revolution with respect to the vernal equinox, the *eclipse year* is a revolution with respect to the moon's ascending node, and the *anomalistic year* is a revolution with respect to the solar perigee.

14. A *calendar year* consists of an integral number of mean solar days and may be a *common year* of 365 days or a *leap year* of 366 days, these years being selected according to the calendars described below so that the average length will agree as nearly as practicable with the length of the tropical year which fixes the periodic changes in the seasons. The average length of the calendar year by the Julian calendar is exactly 365.25 days and by the Gregorian calendar 365.2425 days and these may be designated respectively as a *Julian year* and a *Gregorian year*.

15. The two principal kinds of calendars in use by most of the civilized world since the beginning of the Christian era are the Julian and the Gregorian calendars, the latter being the modern calendar in which the dates are sometimes referred to as "new style" to distinguish them from the dates of the older calendars. Prior to the year 45 B. C. there was more or less confusion in the calendars, intercalations of months and days being arbitrarily made by the priesthood and magistrates to bring the calendar into accord with the seasons and for other purposes.

16. The Julian calendar received its name from Julius Cæsar, who introduced it in the year 45 B. C. This calendar provided that the common year should consist of 365 days and every fourth year of 366 days, each year to begin on January 1. As proposed by Julius Cæsar, the 12 months beginning with January were to be alternately 31 days and 30 days in length with the exception that February should have only 29 days in the common years. When Augustus succeeded Julius Cæsar a few years later, he slightly modified this arrangement by transferring one day from February to the month of Sextilis, or August as it was then renamed, and also transferred the 31st day of September and November to October and December to avoid having three 31-day months in succession.

17. The Gregorian calendar received its name from Pope Gregory, who introduced it in the year 1582. It was immediately adopted by the Catholic countries but was not accepted by England until 1752. This calendar differs from the Julian calendar in having the century years not exactly divisible by 400 to consist of only 365 days, while in the Julian calendar every century year as well as every other year divisible by 4 is taken as a leap year with 366 days. For dates before Christ the year number must be diminished by 1 before testing its divisibility by 4 or 400 since the year 1 B. C. corresponds to the year 0 A. D. The Gregorian calendar will gain on the Julian calendar three days in each 400 years. When originally adopted, in order to adjust the Gregorian calendar so that the vernal equinox should fall upon March 21, as it had at the time of the Council of Nice in 325 A. D., 10 days were dropped and it was ordered that the day following October 4, 1582 of the Julian calendar should be designated as October 15, 1582 of the Gregorian calendar. This difference of 10 days between the dates of the two calendars continued until 1700, which was a leap year according to the Julian calendar and a common year by the Gregorian calendar. The difference between the two then became 11 days and in 1800 was increased to 12 days. Since 1900 the difference has been 13 days and will remain the same until the year 2100.

18. Dates of the Christian era prior to October 4, 1582, will, in general, conform to the Julian calendar. Since that time both calendars have been used. The Gregorian calendar was adopted in England by an act of Parliament passed in 1751, which provided that the day following September 2, 1752, should be called September 14, 1752, and also that the year 1752 and subsequent years should commence on the 1st day of January. Previous to this the legal year in England commenced on March 25. Except for this arbitrary beginning of the year, the old English calendar was the same as the Julian calendar. When Alaska was purchased from Russia by the United States, its calendar was altered by 11 days, one of these days being necessary because of the difference between the Asiatic and American dates when compared across the one hundred and eightieth meridian. Dates in the tables at the back of this volume refer to the Gregorian calendar.

19. The three great circles formed by the intersections of the planes of the earth's equator, the ecliptic, and the moon's orbit with the celestial sphere are represented in figure 1. These circles intersect in six points, three of them being marked by symbols in the figure, namely, the *vernal equinox* ♈ at the intersection of the celestial equator and ecliptic, the ascending *lunar node* ☊ at the intersection of the ecliptic and the projection of the moon's orbit, and the *lunar intersection A* at the intersection of the celestial equator and the projection of the moon's orbit. For brevity these three points are sometimes called respectively "the equinox," "the node," and "the intersection." The vernal equinox, although subject to a slow westward motion of about 50″ per year, is generally taken as a fixed point of reference for the motion of other parts of the solar system. The moon's node has a westward motion of about 19° a year, which is sufficient to carry it entirely around a great circle in a little less than 19 years.

20. The angle ω between the ecliptic and the celestial equator is known as the obliquity of the ecliptic and has a nearly constant value of 23½°. The angle i between the ecliptic and the plane of the moon's orbit is also constant with a value of about 5°.

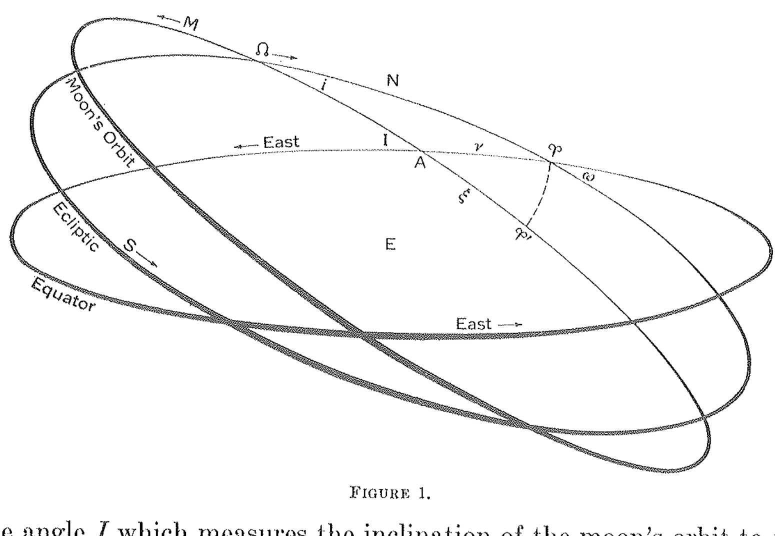

FIGURE 1.

The angle I which measures the inclination of the moon's orbit to the celestial equator might appropriately be called the obliquity of the moon's orbit. Its magnitude changes with the position of the moon's node. When the moon's ascending node coincides with the vernal equinox, the angle I equals the sum of ω and i, or about 28½°, and when the descending node coincides with the vernal equinox, the angle I equals the difference between ω and i, or about 18½°. This variation in the obliquity of the moon's orbit with its period of approximately 18.6 years introduces an important inequality in the tidal movement which must be taken into account.

21. In the celestial sphere the terms "latitude" and "longitude" apply especially to measurements referred to the ecliptic and vernal equinox, but the terms may with propriety also be applied to measurements referred to other great circles and origins, provided they are sufficiently well defined to prevent any ambiguity. For example, we may say "longitude in the moon's orbit measured from the moon's node." Celestial longitude is always understood to be measured toward the east entirely around the circle. Longitude in the celestial equator reckoned from the vernal equinox is called right ascension, and the angular distance north or south of the celestial equator is called declination.

22. The true longitude of any point referred to any great circle in the celestial sphere may be defined as the arc of that circle intercepted between the accepted origin and the projection of the point on the circle, the measurement being always eastward from the origin to the projection of the point. The true longitude of any point will generally be different when referred to different circles, although reckoned from a common origin; and the longitude of a body moving at a uniform rate of speed in one great circle will not have a uniform rate of change when referred to another great circle.

23. The mean longitude of a body moving in a closed orbit and referred to any great circle may be defined as the longitude that would be attained by a point moving uniformly in the circle of reference at the same average angular velocity as that of the body and with the initial position of the point so taken that its mean longitude would be the same as the true longitude of the body at a certain selected position of that body in its orbit. With a common initial point, the mean longitude of a moving body will be the same in whatever circle it may be reckoned. Longitude in the ecliptic and in the celestial equator are usually reckoned from the vernal equinox ♈, which is common to both circles. In order to have an equivalent origin in the moon's orbit, we may lay off an arc ☊ ♈′ (fig. 1) in the moon's orbit equal to ☊ ♈ in the ecliptic and for convenience call the point ♈′ the referred equinox. The mean longitude of any body, if reckoned from either the equinox or the referred equinox, will be the same in any of the three orbits represented. This will, of course, not be the case for the true longitude.

24. Let us now examine more closely the spherical triangle ☊ ♈ A in figure 1. The angles ω and i are very nearly constant for long periods of time and have already been explained. The side ☊ ♈, usually designated by N, is the longitude of the moon's node and is undergoing a constant and practically uniform change due to the regression of the moon's nodes. This westward movement of the node, by which it is carried completely around the ecliptic in a period of approximately 18.6 years, causes a constant change in the form of the triangle, the elements of which are of considerable importance in the present discussion. The value of the angle I, the supplement of the angle ☊ A ♈, has an important effect upon both the range and time of the tide, which will be noted later. The side A ♈, designated by ν, is the right ascension or longitude in the celestial equator of the intersection A. The arc designated by ξ is equal to the side ☊ ♈ − side ☊ A and is the longitude in the moon's orbit of the intersection A. Since the angles i and ω are assumed to be constant, the values of I, ν, and ξ will depend directly upon N, the longitude of the moon's node, and may be readily obtained by the ordinary solution of the spherical triangle ☊ ♈ A. Table 6 give the values of I, ν, and ξ for each degree of N. In the computation of this table the value of ω for the beginning of the twentieth century was used. However, the secular change in the obliquity of the ecliptic is so slow that a difference of a century in the epoch taken as the basis of the computation would have resulted in differences of less than 0.02 of a degree in the tabular values. The table may therefore be used without material error for reductions pertaining to any modern time.

25. Looking again at figure 1, it will be noted that when the longitude of the moon's node is zero the value of the inclination I will equal the sum of ω and i and will be at its maximum. In this position the northern portion of the moon's orbit will be north of the ecliptic. When the longitude of the moon's node is 180°, the moon's orbit will be between the Equator and ecliptic, and the angle I will be equal to angle $\omega-$angle i. The angle I will be always positive and will vary from $\omega-i$ to $\omega+i$. When the longitude of the moon's node equals zero or 180°, the values of ν and ξ will each be zero. For all positions of the moon's node north of the Equator as its longitude changes from 180 to 0°, ν and ξ will have positive values, as indicated in the figure, these arcs being considered as positive when reckoned eastward from ♈ and ♈′, respectively. For all positions of the node south of the Equator, as the longitude changes from 360 to 180°, ν and ξ will each be negative, since the intersection A will then lay to the westward of ♈ and ♈′

DEGREE OF APPROXIMATION

26. The problem of finding expressions for tidal forces and the equilibrium height of the tide in terms of time and place does not admit of a strict solution, but approximate expressions can be obtained which may be carried to as high an order of precision as desired. In ordinary numerical computations exact results are seldom obtained, the degree of precision depending upon the number of decimal places used in the computations, which, in turn, will be determined largely by the magnitude of the quantity sought. In general, the degree of approximation to the value of any quantity expressed numerically will be determined by the number of significant figures used. With a quantity represented by a single significant figure, the error may be as great as 33⅓ percent of the quantity itself, while the use of two significant figures will reduce the maximum error to less than 5 percent of the true value of the quantity. The large possible error in the first case renders it of little value, but in the latter case the approximation is sufficiently close to be useful when only rough results are necessary. The distance of the sun from the earth is popularly expressed by two significant figures as 93,000,000 miles.

27. With three or four significant figures fairly satisfactory approximations may be represented, and with a greater number very precise results may be expressed. For theoretical purposes the highest attainable precision is desirable, but for practical purposes, because of the increase in the labor without a corresponding increase in utility, it will be usually found advantageous to limit the degree of precision in accordance with the prevailing conditions.

28. Frequently a quantity that is to be used as a factor in an expression may be expanded into a series of terms. If the approximate value of such a series is near unity, terms which would affect the third decimal place, if expressed numerically, should usually be retained. The retention of the smaller terms will depend to some extent upon the labor involved since their rejection would not seriously affect the final results.

29. The formulas for the moon's true longitude and parallax on pages 19–20 are said to be given to the second order of approximation, a fraction of the first order being considered as one having an approximate value of 1/20 or 0.05, a fraction of the second order having an approximate value of $(0.05)^2$ or 0.0025, a fraction of third order having an approximate value of $(0.05)^3$ or 0.000125, etc. As these formulas provide important factors in the development of the equations representing the tide-producing forces, they determine to a large extent the degrees of precision to be expected in the results.

DEVELOPMENT OF TIDE-PRODUCING FORCE

FUNDAMENTAL FORMULAS

30. The tide-producing forces exerted by the moon and sun are similar in their action and mathematical expressions obtained for one may therefore by proper substitutions be adapted to the other. Because of the greater importance of the moon in its tide-producing effects, the following development will apply primarily to that body, the necessary changes to represent the solar tides being afterwards indicated.

31. The tide-producing force of the moon is that portion of its gravitational attraction which is effective in changing the water level on the earth's surface. This effective force is the difference between the attraction for the earth as a whole and the attraction for the different particles which constitute the yielding part of the earth's surface; or, if the entire earth were considered to be a plastic mass, the tide-producing force at any point within the mass would be the force that tended to change the position of a particle at that point relative to a particle at the center of the earth. That part of the earth's surface which is directly under the moon is nearer to that body than is the center of the earth and is therefore more strongly attracted since the force of gravity varies inversely as the square of the distance. For the same reason the center of the earth is more strongly attracted by the moon than is that part of the earth's surface which is turned away from the moon.

32. The tide-producing force, being the difference between the attraction for particles situated relatively near together, is small compared with the attraction itself. It may be interesting to note that, although the sun's attraction on the earth is nearly 200 times as great as that of the moon, its tide-producing force is less than one-half that of the moon. If the forces acting upon each particle of the earth were equal and parallel, no matter how great those forces might be, there would be no tendency to change the relative positions of those particles, and consequently there would be no tide-producing force.

33. The tide-producing force may be graphically represented as in figure 2.

Let O= the center of the earth,
C= the center of the moon,
P= any point within or on the surface of the earth.

Then OC will represent the direction of the attractive force of the moon upon a particle at the center of the earth and PC the direction of the attractive force of the moon upon a particle at P. Now, let the magnitude of the moon's attraction at P be represented by the length of the line PC. Then, since the attraction of gravitation varies inversely as the square of the distance, it is necessary, in order to represent the attraction at O on the same scale, to take a line CQ of such length that $CQ : CP = \overline{CP}^2 : \overline{CO}^2$.

34. The line PQ, joining P and Q, will then represent the direction and magnitude of the resultant force that tends to disturb the position of P relative to O, for it represents the difference between the force PC and a force through P equal and parallel to the force QC which acts upon O. This last statement may be a little clearer to the reader if he will consider the force PC as being resolved into a force PD equal and parallel to QC, and the force PQ. The force PD, acting upon the particle at P, being equal and parallel to the force QC, acting upon a particle at O, will have no tendency to change the position of P relative to O. The remaining force PQ will tend to alter the position of P relative to O and is the tide-producing force of the moon at P. The force PQ may be resolved into a vertical component PR, which tends to raise the water at P, and the horizontal component PT, which tends to move the water horizontally.

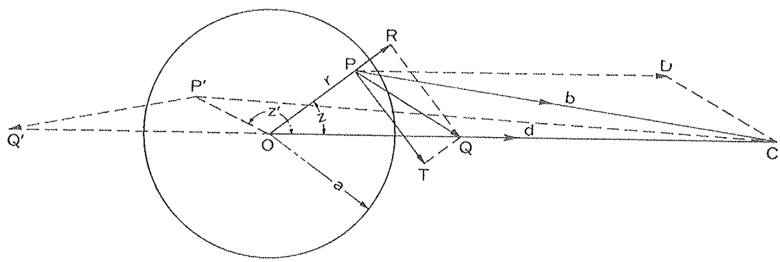

FIGURE 2.

35. If the point P' is taken so that the distance CP' is greater than the distance CO, the tide-producing force $P'Q'$ will be directed away from the moon. While at first sight this may appear paradoxical, it will be noted that the moon tends to separate O from P', but as O is taken as the point of reference, this resulting force that tends to separate the points is considered as being applied at the point P' only.

36. To express the tide-producing force by mathematical equations, refer to figure 2 and let

$r = OP$ = distance of particle P from center of earth,
$b = PC$ = distance of particle P from center of moon,
$d = OC$ = distance from center of earth to center of moon,
$z = COP$ = angle at center of earth between OP and OC.

Also let

M = mass of moon,
E = mass of earth,
a = mean radius of earth,
μ = attraction of gravitation between unit masses at unit distance.
g = mean acceleration of gravity on earth's surface.

Since the force of gravitation varies directly as the mass and inversely as the square of the distance,

$$\text{Attraction of moon for unit mass at point } O \text{ in direction } OC = \frac{\mu M}{d^2} \quad (2)$$

Attraction of moon for unit mass at point P in direction $PC=\frac{\mu M}{b^2}$ (3)

37. Let each of these forces be resolved into a vertical component along the radius OP and a horizontal component perpendicular to the same in the plane OPC, and consider the direction from O toward P as positive for the vertical component and the direction corresponding to the azimuth of the moon as positive for the horizontal component. We then have from (2) and (3)

$$\text{Attraction at } O \text{ in direction } O \text{ to } P=\frac{\mu M}{d^2}\cos z \qquad (4)$$

$$\text{Attraction at } O \text{ perpendicular to } OP=\frac{\mu M}{d^2}\sin z \qquad (5)$$

$$\text{Attraction at } P \text{ in direction } O \text{ to } P=\frac{\mu M}{b^2}\cos CPR \qquad (6)$$

$$\text{Attraction at } P \text{ perpendicular to } OP=\frac{\mu M}{b^2}\sin CPR \qquad (7)$$

38. The tide-producing force of the moon at any point P is measured by the difference between the attraction at P and at the center of the earth. Letting

F_v=vertical component of tide-producing force, and
F_a=horizontal component in azimuth of moon,

and taking the differences between (6) and (4) and between (7) and (5), we obtain the following expressions for these component forces in terms of the unit μ:

$$F_v/\mu=M\left(\frac{\cos CPR}{b^2}-\frac{\cos z}{d^2}\right) \qquad (8)$$

$$F_a/\mu=M\left(\frac{\sin CPR}{b^2}-\frac{\sin z}{d^2}\right) \qquad (9)$$

39. From the plane triangle COP the following relations may be obtained:

$$b^2=r^2+d^2-2rd\cos z=d^2[1-2(r/d)\cos z+(r/d)^2] \qquad (10)$$

$$\sin CPR=\sin CPO=(d/b)\sin z=\frac{\sin z}{[1-2(r/d)\cos z+(r/d)^2]^{\frac{1}{2}}} \qquad (11)$$

$$\cos CPR=(1-\sin^2 CPR)^{\frac{1}{2}}=\frac{\cos z-r/d}{[1-2(r/d)\cos z+(r/d)^2]^{\frac{1}{2}}} \qquad (12)$$

40. In figure 2 it will be noted that the value of z, being reckoned in any plane from the line OC, may vary from zero to 180°, and also that the angle CPR increases as z increases within the same limits. Sin z and sin CPR will therefore always be positive. As the angle OCP is always very small, the angle CPR will differ by only a very small amount from the angle z and will usually be in the same quadrant. In obtaining the square root for the numerator of (12) it was therefore necessary to use only that sign which would preserve this relationship. The denominators of (11) and (12) are to be considered as positive.

41. Substituting in equations (8) and (9) the equivalents for b, sin CPR, and cos CPR from equations (10) to (12), the following basic formulas are obtained for the vertical and horizontal components of the tide-producing force at any point P at r distance from the center of the earth:

$$F_v/\mu=\frac{M}{d^2}\left[\frac{\cos z-r/d}{\{1-2(r/d)\cos z+(r/d)^2\}^{\frac{3}{2}}}-\cos z\right] \qquad (13)$$

$$F_a/\mu=\frac{M}{d^2}\left[\frac{\sin z}{\{1-2(r/d)\cos z+(r/d)^2\}^{\frac{3}{2}}}-\sin z\right] \qquad (14)$$

42. To express these forces in their relation to the mean acceleration of gravity on the earth's surface, represented by the symbol g, we have

$$g/\mu=E/a^2, \quad \text{or} \quad \mu/g=a^2/E \qquad (15)$$

in which E is the mass and a is the mean radius of the earth. Substituting the above in formulas (13) and (14), we may write

$$F_v/g=(M/E)\ (a/d)^2\left[\frac{\cos z-r/d}{\{1-2(r/d)\ \cos\ z+(r/d)^2\}^{\frac{3}{2}}}=\cos z\right] \qquad (16)$$

$$F_a/g=(M/E)\ (a/d)^2\left[\frac{\sin z}{\{1-2(r/d)\ \cos\ z+(r/d)^2\}^{\frac{3}{2}}}-\sin z\right] \qquad (17)$$

43. Formulas (16) and (17) represent completely the vertical and horizontal components of the lunar tide-producing force at any point in the earth. If r is taken equal to the mean radius a, the formulas will involve the constant ratio M/E and two variable quantities—the angle z which is the moon's zenith distance, and the ratio a/d which is the sine of the moon's horizontal parallax in respect to the mean radius of the earth. Because of the smallness of the ratio a/d it may also be taken as the parallax itself expressed as a fraction of a radian. The parallax is largest when the moon is in perigee and at this time the tide-producing force will reach its greatest magnitude. A more rapid change in the tidal force at any point on the earth's surface is caused by the continuous change in the zenith distance of the moon resulting from the earth's rotation. The vertical component attains its maximum value when z equals zero, and the horizontal component has its maximum value when z is a little less than 45°. Substituting numerical values in formulas (16) and (17) and in similar formulas for the tide-producing force of the sun, the following are obtained as the approximate extreme component forces when the moon and sun are nearest the earth:

$$\text{Greatest } F_v/g=.144\times10^{-6} \text{ for moon, or } .054\times10^{-6} \text{ for sun} \qquad (18)$$

$$\text{Greatest } F_a/g=.107\times10^{-6} \text{ for moon, or } .041\times10^{-6} \text{ for sun} \qquad (19)$$

The horizontal component of the tide-producing force may be measured by its deflection of the plumb line, the relation of this component to gravity as expressed by the above formula being the tangent of the angle of deflection. Under the most favorable conditions the greatest deflection due to the moon is about 0.022″ and the greatest deflection due to the sun is less than 0.009″ of arc.

44. To simplify the preceding formulas, the quantity involving the fractional exponent may be developed by Maclaurin's theorem into a series arranged according to the ascending powers of r/d, this being a small fraction with an approximate maximum value of 0.018. Thus

$$\frac{1}{\{1-2(r/d)\ \cos\ z+(r/d)^2\}^{\frac{3}{2}}}=1+3\ \cos\ z\ (r/d)$$
$$+3/2\ (5\ \cos^2 z-1)(r/d)^2$$
$$+5/2\ (7\ \cos^3 z-3\ \cos\ z)(r/d)^3+\text{etc.} \qquad (20)$$

45. Substituting (20) in formulas (16) and (17) and neglecting the higher powers of r/d, we obtain the following formulas:

$$F_v\ /g=3\ (M/E)(a/d)^2\ (\cos^2 z-1/3)\ (r/d)$$
$$+3/2\ (M/E)(a/d)^2\ (5\ \cos^3 z-3\ \cos\ z)\ (r/d)^2 \qquad (21)$$

$$F_a\ /g=3/2\ (M/E)(a/d)^2\ (\sin 2\ z)\ (r/d)$$
$$+3/2\ (M/E)(a/d)^2 \sin z\ (5\ \cos^2 z-1)\ (r/d)^2 \qquad (22)$$

46. If r, which represents the distance of the point of observation for the center of the earth, is replaced by the mean radius a, it will be noted that the first term of each of the above formulas involves the cube of the ratio a/d while the second term involves the fourth power of this quantity. This ratio is essentially the moon's parallax expressed in the radian unit. These terms may now be written as separate formulas and for convenience of identification the digits "3" and "4" will be annexed to the formula symbol to represent respectively the terms involving the cube and fourth power of the parallax. Thus

$$F_{v3}\ /g=3\ (M/E)(a/d)^3(\cos^2 z-1/3) \qquad (23)$$

$$F_{v4}\ /g=3/2\ (M/E)(a/d)^4(5\ \cos^3 z-3\ \cos\ z) \qquad (24)$$

$$F_{a3}\ /g=3/2\ (M/E)(a/d)^3 \sin 2z \qquad (25)$$

$$F_{a4}\ /g=3/2\ (M/E)(a/d)^4 \sin z\ (5\ \cos^2 z-1) \qquad (26)$$

Formulas (23) and (25) involving the cube of the parallax represent the principal part of the tide-producing force. For the moon this is about 98 per cent of the whole and for the sun a higher percentage. The part of the tide-producing force represented by formulas (24) and (26) and involving the fourth power of the parallax is of very little practical importance but as a matter of theoretical interest will be later given further attention.

47. An examination of formulas (23) and (25) shows that the principal part of the tide-producing force is symmetrically distributed over the earth's surface with respect to a plane through the center of the earth and perpendicular to a line joining the centers of the earth and moon. The vertical component (23) has a maximum positive value when the zenith distance $z=0$ or 180° and a maximum negative value when $z=90°$, the maximum negative value being one-half as great as the maximum positive value. The vertical component becomes zero when $z = \cos^{-1} \pm \sqrt{1/3}$ (approx. 54.74° and 125.26°). The horizontal component (25) has its maximum value when $z = 45°$ and an equal maximum negative value when $z = 135°$. The horizontal component becomes zero when $z = 0$, 90°, or 180.

48. If numerical values applicable to the mean parallax of the moon are substituted in (23) and (25), these component forces may be written

$$F_{v3}/g \text{ at mean parallax} = 0.000,000,167\ (\cos^2 z - 1/3) \qquad (27)$$

$$F_{a3}/g \text{ at mean parallax} = 0.000,000,084 \sin 2z \qquad (28)$$

For the corresponding components of the solar tide-producing force, the numerical coefficients will be 0.46 times as great as those in the above formulas.

49. For the extreme values of the components represented by (23) and (25), with the moon and sun nearest the earth, the following may be obtained by suitable substitutions:

$$\text{Greatest } F_{v3}/g = .140 \times 10^{-6} \text{ for moon, or } .054 \times 10^{-6} \text{ for sun} \qquad (29)$$

$$\text{Greatest } F_{a3}/g = .105 \times 10^{-6} \text{ for moon, or } .041 \times 10^{-6} \text{ for sun} \qquad (30)$$

Comparing the above with (18) and (19), it will be noted that the maximum values of the lunar components involving the cube of the moon's parallax are only slightly less than the corresponding maximum values for the entire lunar force, while for the solar components the differences are too small to be shown with the number of decimal places used.

VERTICAL COMPONENT OF FORCE

50. It is now proposed to expand into a series of harmonic terms formula (23) which represents the principal vertical component of the lunar tide-producing force. In figure 3 let O represent the center of the earth and let projections on the celestial sphere be as follows:

C,	the north pole
$I\,M'\,P'$,	the earth's equator
$I\,M$,	the moon's orbit
M,	the position of the moon
P,	the place of observation
$C\,M\,M'$,	the hour circle of the moon
$C\,P\,P'$,	the meridian of place of observation
I,	the intersection of moon's orbit and equator

Also let

I = angle $M\,I\,M'$ = inclination of moon's orbit to earth's equator
t = arc $P'\,M'$ or angle PCM = hour angle of moon
$X = IP'$ = longitude of P measured in celestial equator from intersection I
$j = IM$ = longitude of moon in orbit reckoned from intersection I
$z = PM$ = zenith distance of moon
$D = M'M$ = declination of moon
$Y = P'P$ = latitude of P

The solution of a number of the spherical triangles represented in figure 3 will provide certain relations needed in the development of the formulas for the tide-producing force.

51. In spherical triangle MCP, the angle C equals t and the sides MC and PC are the complements of D and Y, respectively. We may therefore write

$$\cos z = \sin Y \sin D + \cos Y \cos D \cos t \qquad (31)$$

Substituting this value in formula (23), we obtain

$$\begin{aligned} F_{v3}/g = {} & 3/2\ (M/E)(a/d)^3(1/2 - 3/2 \sin^2 Y)(2/3 - 2\sin^2 D) \ldots\ldots F_{v30}/g \\ & + 3/2\ (M/E)(a/d)^3 \sin 2Y \sin 2D \cos t \ldots\ldots F_{v31}/g \\ & + 3/2\ (M/E)(a/d)^3 \cos^2 Y \cos^2 D \cos 2t \ldots\ldots F_{v32}/g \qquad (32) \end{aligned}$$

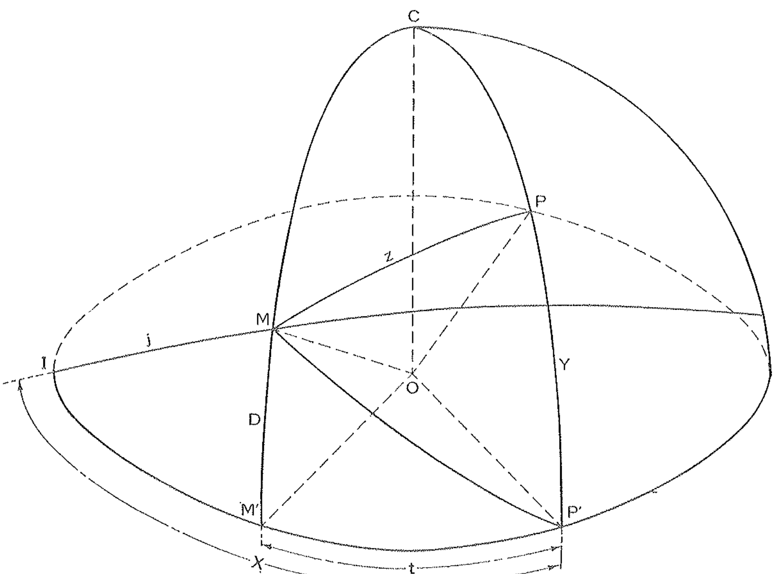

FIGURE 3.

52. In formula (32) the vertical component of the tide-producing force has been separated into three parts. The first term is independent of the rotation of the earth but is subject to variations arising from changes in declination and distance of the moon. It includes what are known as the *long-period constituents*, that is to say, constituents with periods somewhat longer than a day and in general a half month or longer. The second term involves the cosine of the hour angle (t) of the moon and this includes the *diurnal constituents* with periods approximating the lunar day. The last term involves the cosine of twice the hour angle of the moon and includes the *semidiurnal constituents* with periods approximating the half lunar day. The grouping of the tidal constituents according to their approximate periods affords an important classification in the further development of the tidal forces and these groups will be called *classes* or *species*. Symbols pertaining to a particular species are often identified by a subscript indicating the number of periods in a day, the subscript o being used for the long-period constituents. In formula (32) the individual terms are identified by the annexation of the species subscript to the general symbol for the formula.

53. As written, all of the three terms of formula (32) have the same coefficient $3/2\ (M/E)\ (a/d)^3$. In each case the latitude (Y) factor has a maximum value of unity, this maximum being negative for the first term. For the long-period term (F_{v30}/g), the latitude factor has a maximum positive value of ½ at the equator, becomes zero in latitude 35.26° (approximately), and reaches a maximum negative value of -1 at the poles, the factor being the same for corresponding latitudes in both northern and southern hemispheres. For the diurnal term (F_{v31}/g), the latitude factor is positive for the northern hemisphere and negative for the southern hemisphere. It has a maximum value of unity in latitude 45° and is zero at the equator and poles. For the semidiurnal terms (F_{v32}/g), the latitude factor is always positive and has a maximum value of unity at the equator and equals zero at the poles.

54. For extreme values attainable for the declinational (D) factors, consideration must be given to the greatest declination which can be reached by the tide-producing body. The periodic maximum declination reached by the moon in its 18.6 year node-cycle is 28.6° but this may be slightly increased by other inequalities in the moon's motion. The maximum declination for the sun, taken the same as the obliquity of the ecliptic, is 23.45°. The declination factor of the long-period term (F_{v30}/g) has a maximum value of 2/3 when the declination is zero. It diminishes with increasing north or south declination but must always remain positive because of the limits of the declination. For the diurnal term (F_{v31}/g) the declinational factor has its greatest value when the declination is greatest. For the moon the maximum value of this factor is approximately 0.841 and for the sun 0.730. This factor is positive for the north declination and negative for the south declination. For the semidiurnal term (F_{v32}/g) the declinational factor for both moon and sun is always positive and has a maximum value of unity at zero declination.

55. The greatest numerical values for the several terms of the vertical component of the tide-producing force as represented by formula (32) and applicable to the time when the moon and sun are nearest the earth, are as follows:

$$\text{Greatest } F_{v30}/g = -.070\times10^{-6} \text{ for moon, or } -.027\times10^{-6} \text{ for sun} \quad (33)$$
$$\text{Greatest } F_{v31}/g = \pm.088\times10^{-6} \text{ for moon, or } \pm.030\times10^{-6} \text{ for sun} \quad (34)$$
$$\text{Greatest } F_{v32}/g = +.105\times10^{-6} \text{ for moon, or } +.041\times10^{-6} \text{ for sun} \quad (35)$$

For the long-period term (33) the greatest value applies to either pole and is negative. For the diurnal term (34) the greatest value applies in latitude 45° and may be positive or negative according to whether the latitude and declinational factors have the same or opposite signs. For the semidiurnal term (35) the greatest value applies to the equator and is positive.

56. Referring to formula (32), let a/c equal the mean value of parallax a/d. Then a/d may be replaced by its equivalent $(a/c)\ (c/d)$, in which the fraction c/d expresses the relation between the true and the mean parallax. Also let $U=(M/E)\ (a/c)^3$, the numerical value of which will be found in table 1. Expressing separately the three terms of formula (32), we then have

$$F_{v30}/g=3/2\ U\ (c/d)^3\ (1/2-3/2\sin^2 Y)(2/3-2\sin^2 D) \tag{36}$$
$$F_{v31}/g=3/2\ U\ (c/d)^3 \sin 2Y \sin 2D \cos t \tag{37}$$
$$F_{v32}/g=3/2\ U\ (c/d)^3 \cos^2 Y \cos^2 D \cos 2t \tag{38}$$

57. Referring to figure 3, the following relations may be obtained from the right spherical triangles MIM' and $MP'M'$ and the oblique spherical triangle $MP'I$:

$$\sin D=\sin I \sin j \tag{39}$$
$$\cos D \cos t=\cos MP' \tag{40}$$
$$\cos MP'=\cos X \cos j+\sin X \sin j \cos I \tag{41}$$
$$\cos D \cos t=\cos X \cos j+\sin X \sin j \cos I$$
$$=\cos^2 \tfrac{1}{2}I \cos (X-j)+\sin^2 \tfrac{1}{2}I \cos (X+j) \tag{42}$$

58. Replacing the functions of D and t in formulas (36) to (38) by their equivalents derived from equations (39) and (42), there are obtained the following:

$$F_{v30}/g=3/2\ U(c/d)^3(1/2-3/2\sin^2 Y)\times$$
$$[2/3-\sin^2 I+\sin^2 I \cos 2j] \tag{43}$$
$$F_{v31}/g=3/2\ U(c/d)^3 \sin 2Y\times$$
$$[\sin I \cos^2 \tfrac{1}{2}I \cos (X+90^\circ-2j)$$
$$+1/2 \sin 2I \cos (X-90^\circ)$$
$$+\sin I \sin^2 \tfrac{1}{2}I \cos (X-90^\circ+2j)] \tag{44}$$
$$F_{v32}/g=3/2\ U(c/d)^3 \cos^2 Y\times$$
$$[\cos^4 \tfrac{1}{2}I \cos (2X-2j)$$
$$+1/2 \sin^2 I \cos 2X$$
$$+\sin^4 \tfrac{1}{2}I \cos (2X+2j)] \tag{45}$$

The above formulas involve the moon's actual distance d and its true longitude j as measured in its orbit from the intersection. While these are functions of time, they do not vary uniformly because of certain inequalities in the motion of the moon, and it is now desired to replace these quantities by elements that do change uniformly.

59. Referring to paragraphs 23–24 and to figure 1, it will be noted that longitude measured from intersection A in the moon's orbit equals the longitude measured from the referred equinox ♈′ less arc ξ, and longitude measured from intersection A in the celestial equator equals the longitude measured from the equinox ♈ less arc ν.
Now let

s' = true longitude of moon in orbit referred to equinox
s = mean longitude of moon referred to equinox
k = difference $(s'-s)$

Then

$$j=s'-\xi=s-\xi+k \tag{46}$$

60. In figure 4 let S' and P' be the points where the hour circles of the mean sun and place of observation intersect the celestial equator, ♈ the vernal equinox, and I the lunar intersection. Then X will equal the arc $P'I$ and ν the arc I♈. Now let

h = mean longitude of sun
T = hour angle of mean sun

Then

$$X=T+h-\nu \tag{47}$$

61. Substituting the values of j and X from (46) and (47) in formulas (43) to (45), these may be written

$$F_{v30}/g=3/2\ U(1/2-3/2\sin^2 Y)\times$$
$$[(c/d)^3(2/3-\sin^2 I)$$
$$+(c/d)^3\sin^2 I\cos(2s-2\xi+2k)] \qquad (48)$$

$$F_{v31}/g=3/2\ U\sin 2Y\times$$
$$[(c/d)^3\sin I\cos^2\tfrac{1}{2}I\cos(T-2s+h+2\xi-\nu+90^\circ-2k)$$
$$+1/2\ (c/d)^3\sin 2I\cos(T+h-\nu-90^\circ)$$
$$+(c/d)^3\sin I\sin^2\tfrac{1}{2}I\cos(T+2s+h-2\xi-\nu-90^\circ+2k)] \qquad (49)$$

$$F_{v32}/g=3/2\ U\cos^2 Y\times$$
$$[(c/d)^3\cos^4\tfrac{1}{2}I\cos(2T-2s+2h+2\xi-2\nu-2k)$$
$$+1/2\ (c/d)^3\sin^2 I\cos(2T+2h-2\nu)$$
$$+(c/d)^3\sin^4\tfrac{1}{2}I\cos(2T+2s+2h-2\xi-2\nu+2k) \qquad (50)$$

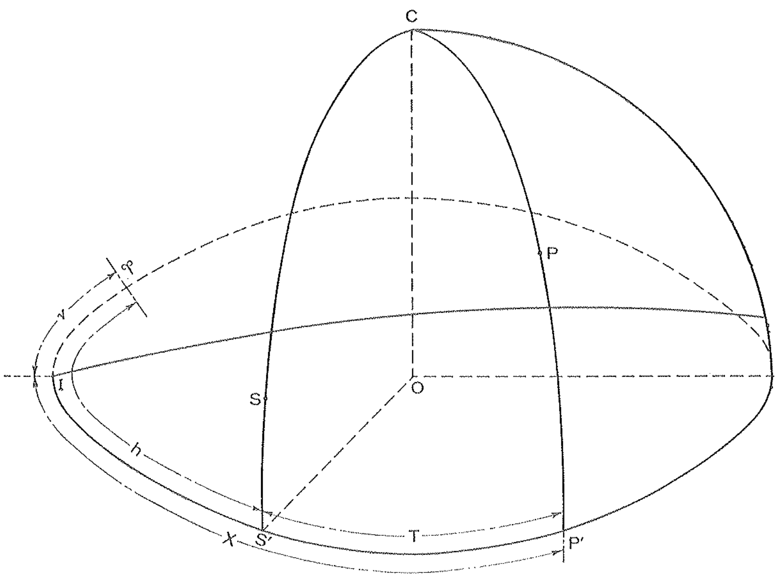

FIGURE 4.

Disregarding at this time the slow change in the function of I, the variable part of each term of the above formulas may be expressed in one of the following forms—$(c/d)^3$, $(c/d)^3\cos A$, $(c/d)^3\cos(A+2k)$, or $(c/d)^3\cos(A-2k)$, in which A includes all the elements of the variable angular function excepting the multiple of k.

62. The following equations for the motion of the moon were adapted from Godfray's Elementary Treatise on the Lunar Theory:

s' = true longitude of moon (in radians)

$=s$ ---------------------------------- (mean longitude)

$+2e\sin(s-p)+5/4\ e^2\sin 2(s-p)$ ------ (elliptic inequality)

$+15/4\ me\sin(s-2h+p)$ ----------- (evectional inequality)

$+11/8\ m^2\sin 2(s-h)$ ---------- (variational inequality) (51)

$$\begin{aligned} c/d &= \text{(true parallax of moon)/(mean parallax of moon)} \\ &= \text{unity} \\ &+e \cos (s-p)+e^2 \cos 2(s-p) \ldots\ldots\ldots\ldots \text{(elliptic inequality)} \\ &+15/8\ me \cos (s-2h+p) \ldots\ldots\ldots\ldots \text{(evectional inequality)} \\ &+m^2 \cos 2(s-h) \ldots\ldots\ldots\ldots\ldots \text{(variational inequality)} \qquad (52) \end{aligned}$$

in which

s' = true longitude of moon in orbit (referred to equinox)
s = mean longitude of moon
h = mean longitude of sun
p = mean longitude of lunar perigee
e = eccentricity of moon's orbit = 0.0549
m = ratio of mean motion of sun to that of moon = 0.0748

The elements e and m are small fractions of the first order and the square of either or the product of both may be considered as being of the second order. In the following development the higher powers of these elements will be omitted.

63. Since k has been taken as the difference between the true and the mean longitude of the moon, we may obtain from (51)

$$\begin{aligned} k &= 2e \sin (s-p)+5/4\ e^2 \sin 2(s-p) \\ &\quad +15/4\ me \sin (s-2h+p)+11/8\ m^2 \sin 2(s-h) \qquad (53) \end{aligned}$$

The value of k is always small, its maximum value being about 0.137 radian. It may therefore be assumed without material error that the sine of k or the sine of $2k$ is equal to the angle itself. Then

$$\begin{aligned} \sin 2k = 2k &= 4e \sin (s-p)+5/2\ e^2 \sin 2(s-p) \\ &\quad +15/2\ me \sin (s-2h+p)+11/4\ m^2 \sin 2(s-h) \qquad (54) \end{aligned}$$

$$\begin{aligned} \cos 2k &= 1-2 \sin^2 k = 1-2k^2 \\ &= 1-4e^2+4e^2 \cos 2(s-p) \qquad (55) \end{aligned}$$

terms smaller than those of the second order being omitted.

64. Cubing (52) and neglecting the smaller terms, we obtain

$$\begin{aligned} (c/d)^3 &= 1+3/2\ e^2+3e \cos (s-p)+9/2\ e^2 \cos 2(s-p) \\ &\quad +45/8\ me \cos (s-2h+p)+3\ m^2 \cos 2(s-h) \qquad (56) \end{aligned}$$

Multiplying (54) and (55) by (56)

$$\begin{aligned} (c/d)^3 \sin 2k &= 4e \sin (s-p)+17/2\ e^2 \sin 2(s-p) \\ &\quad +15/2\ me \sin (s-2h+p)+11/4\ m^2 \sin 2(s-h) \qquad (57) \end{aligned}$$

$$\begin{aligned} (c/d)^3 \cos 2k &= 1-5/2\ e^2+3\ e \cos (s-p)+17/2\ e^2 \cos 2(s-p) \\ &\quad +45/8\ me \cos (s-2h+p)+3\ m^2 \cos 2(s-h) \qquad (58) \end{aligned}$$

65. From (56), (57), and (58), we may obtain the following general expressions applicable to the further development of formulas (48) to (50). Negative coefficients have been avoided by the introduction of 180° in the angle when necessary.

$$\begin{aligned} &(c/d)^3 \cos (A-2k) = (c/d)^3 \cos 2k \cos A+(c/d)^3 \sin 2k \sin A \\ &= (1-5/2\ e^2) \cos A \\ &+7/2\ e \cos (A-s+p)+1/2e \cos (A+s-p+180^\circ) \\ &+17/2\ e^2 \cos (A-2s+2p) \\ &+105/16\ me \cos (A-s+2h-p)+15/16\ me \cos (A+s-2h+p+180^\circ) \\ &+23/8\ m^2 \cos (A-2s+2h)+1/8\ m^2 \cos (A+2s-2h) \qquad (59) \end{aligned}$$

$$(c/d)^3 \cos A = (1+3/2\ e^2) \cos A$$
$$+3/2\ e \cos (A-s+p)+3/2\ e \cos (A+s-p)$$
$$+9/4\ e^2 \cos (A-2s+2p)+9/4\ e^2 \cos (A+2s-2p)$$
$$+45/16\ me \cos (A-s+2h-p)+45/16\ me \cos (A+s-2h+p)$$
$$+3/2\ m^2 \cos (A-2s+2h)+3/2\ m^2 \cos (A+2s-2h) \qquad (60)$$

$$(c/d)^3 \cos (A+2k) = (c/d)^3 \cos 2k \cos A - (c/d)^3 \sin 2k \sin A$$
$$= (1-5/2\ e^2) \cos A$$
$$+7/2\ e \cos (A+s-p)+1/2\ e \cos (A-s+p+180°)$$
$$+17/2\ e^2 \cos (A+2s-2p)$$
$$+105/16\ me \cos (A+s-2h+p)+15/16\ me \cos (A-s+2h-p+180°)$$
$$+23/8\ m^2 \cos (A+2s-2h)+1/8\ m^2 \cos (A-2s+2h) \qquad (61)$$

66. After suitable substitutions for A have been made in the three preceding equations they are immediately applicable to the final expansion of the several terms in formulas (48) to (50), excepting the first term of (48) for which formula (56) may be used directly. Each term in the expanded formulas given below represents a constituent of the lunar tide-producing force and for convenience of reference is designated by the letter A with a subscript. There are also given the generally recognized symbols for the principal constituents, and when such a symbol is enclosed in brackets it signifies that the term given only partially represents the constituent so named.

67. Formula for long-period constituents of vertical component of principal lunar tide-producing force:

$$F_{v30}\ /g = 3/2\ U(1/2-3/2 \sin^2 Y)\times$$

(A_1) $[(2/3-\sin^2 I)\{(1+3/2\ e^2)$ ---------- permanent term
(A_2) $+3\ e \cos (s-p)$ ---------- Mm
(A_3) $+9/2\ e^2 \cos (2s-2p)$
(A_4) $+45/8\ me \cos (s-2h+p)$
(A_5) $+3\ m^2 \cos (2s-2h)\}$ ---------- MSf
(A_6) $+ \sin^2 I\{(1-5/2\ e^2) \cos (2s-2\xi)$ ---------- Mf
(A_7) $+7/2\ e \cos (3s-p-2\xi)$
(A_8) $+1/2\ e \cos (s+p+180°-2\xi)$
(A_9) $+17/2\ e^2 \cos (4s-2p-2\xi)$
(A_{10}) $+105/16\ me \cos (3s-2h+p-2\xi)$
(A_{11}) $+15/16\ me \cos (s+2h-p+180°-2\xi)$
(A_{12}) $+23/8\ m^2 \cos (4s-2h-2\xi)$
(A_{13}) $+1/8\ m^2 \cos (2h-2\xi)\}]$ (62)

68. Formula for diurnal constituents of vertical component of principal lunar tide-producing force:

$$F_{v31}\ /g = 3/2\ U \sin 2Y\times$$

(A_{14}) $[\sin I \cos^2 1/2I$
$\{(1-5/2\ e^2) \cos (T-2s+h+90°+2\xi-\nu)$ ---- O_1
(A_{15}) $+7/2\ e \cos (T-3s+h+p+90°+2\xi-\nu)$ ---- Q_1
(A_{16}) $+1/2\ e \cos (T-s+h-p-90°+2\xi-\nu)$ ---- $[M_1]$
(A_{17}) $+17/2\ e^2 \cos (T-4s+h+2p+90°+2\xi-\nu)$ -- $2Q_1$
(A_{18}) $+105/16\ me \cos (T-3s+3h-p+90°+2\xi-\nu)$ - ρ_1
(A_{19}) $+15/16\ me \cos (T-s-h+p-90°+2\xi-\nu)$
(A_{20}) $+23/8\ m^2 \cos (T-4s+3h+90°+2\xi-\nu)$ ---- σ_1
(A_{21}) $+1/8\ m^2 \cos (T-h+90°+2\xi-\nu)\}$

(Formula continued next page)

(A_{22})	$+\sin 2I\{(1/2+3/4\ e^2)\cos(T+h-90^\circ-\nu)$	$[K_1]$
(A_{23})	$+3/4\ e\cos(T-s+h+p-90^\circ-\nu)$	$[M_1]$
(A_{24})	$+3/4\ e\cos(T+s+h-p-90^\circ-\nu)$	J_1
(A_{25})	$+9/8\ e^2\cos(T-2s+h+2p-90^\circ-\nu)$	
(A_{26})	$+9/8\ e^2\cos(T+2s+h-2p-90^\circ-\nu)$	
(A_{27})	$+45/32\ me\cos(T-s+3h-p-90^\circ-\nu)$	χ_1
(A_{28})	$+45/32\ me\cos(T+s-h+p-90^\circ-\nu)$	θ_1
(A_{29})	$+3/4\ m^2\cos(T-2s+3h-90^\circ-\nu)$	MP_1
(A_{30})	$+3/4\ m^2\cos(T+2s-h-90^\circ-\nu)\}$	SO_1
	$+\sin I\sin^2 \tfrac{1}{2}I$	
(A_{31})	$\{(1-5/2\ e^2)\cos(T+2s+h-90^\circ-2\xi-\nu)$	OO_1
(A_{32})	$+7/2\ e\cos(T+3s+h-p-90^\circ-2\xi-\nu)$	KQ_1
(A_{33})	$+1/2\ e\cos(T+s+h+p+90^\circ-2\xi-\nu)$	
(A_{34})	$+17/2\ e^2\cos(T+4s+h-2p-90^\circ-2\xi-\nu)$	
(A_{35})	$+105/16\ me\cos(T+3s-h+p-90^\circ-2\xi-\nu)$	
(A_{36})	$+15/16\ me\cos(T+s+3h-p+90^\circ-2\xi-\nu)$	
(A_{37})	$+23/8\ m^2\cos(T+4s-h-90^\circ-2\xi-\nu)$	
(A_{38})	$+1/8\ m^2\cos(T+3h-90^\circ-2\xi-\nu)\}]$	(63)

69. Formula for semidiurnal constituents of vertical component of principal lunar tide-producing force:

$$F_{v32}/g=3/2\ U\cos^2 Y\times$$

(A_{39})	$[\cos^4 \tfrac{1}{2}I\{(1-5/2\ e^2)\cos(2T-2s+2h+2\xi-2\nu)$	M_2
(A_{40})	$+7/2\ e\cos(2T-3s+2h+p+2\xi-2\nu)$	N_2
(A_{41})	$+1/2\ e\cos(2T-s+2h-p+180^\circ+2\xi-2\nu)$	$[L_2]$
(A_{42})	$+17/2\ e^2\cos(2T-4s+2h+2p+2\xi-2\nu)$	$2N_2$
(A_{43})	$+105/16\ me\cos(2T-3s+4h-p+2\xi-2\nu)$	ν_2
(A_{44})	$+15/16\ me\cos(2T-s+p+180^\circ+2\xi-2\nu)$	λ_2
(A_{45})	$+23/8\ m^2\cos(2T-4s+4h+2\xi-2\nu)$	μ_2
(A_{46})	$+1/8\ m^2\cos(2T+2\xi-2\nu)\}$	
(A_{47})	$+\sin^2 I\{(1/2+3/4\ e^2)\cos(2T+2h-2\nu)$	$[K_2]$
(A_{48})	$+3/4\ e\cos(2T-s+2h+p-2\nu)$	$[L_2]$
(A_{49})	$+3/4\ e\cos(2T+s+2h-p-2\nu)$	KJ_2
(A_{50})	$+9/8\ e^2\cos(2T-2s+2h+2p-2\nu)$	
(A_{51})	$+9/8\ e^2\cos(2T+2s+2h-2p-2\nu)$	
(A_{52})	$+45/32\ me\cos(2T-s+4h-p-2\nu)$	
(A_{53})	$+45/32\ me\cos(2T+s+p-2\nu)$	
(A_{54})	$+3/4\ m^2\cos(2T-2s+4h-2\nu)$	
(A_{55})	$+3/4\ m^2\cos(2T+2s-2\nu)\}$	
(A_{56})	$+\sin^4 \tfrac{1}{2}I\{(1-5/2\ e^2)\cos(2T+2s+2h-2\xi-2\nu)$	
(A_{57})	$+7/2\ e\cos(2T+3s+2h-p-2\xi-2\nu)$	
(A_{58})	$+1/2\ e\cos(2T+s+2h+p+180^\circ-2\xi-2\nu)$	
(A_{59})	$+17/2\ e^2\cos(2T+4s+2h-2p-2\xi-2\nu)$	
(A_{60})	$+105/16\ me\cos(2T+3s+p-2\xi-2\nu)$	
(A_{61})	$+15/16\ me\cos(2T+s+4h-p+180^\circ-2\xi-2\nu)$	
(A_{62})	$+23/8\ m^2\cos(2T+4s-2\xi-2\nu)$	
(A_{63})	$+1/8\ m^2\cos(2T+4h-2\xi-2\nu)\}]$	(64)

70. *Arguments.*—Except for the slow changes in the values of I, ξ, and ν which result from the revolution of the moon's node, each term other than the permanent one in the three preceding formulas is an harmonic function of an angle that changes uniformly with time. This angle is known as the *argument* of the constituent, also as the *equilibrium argument* when obtained in connection with the development of the equilibrium tide. By analogy, the argument of the permanent term may be considered as zero, the cosine of zero being unity.

71. The argument serves to identify the constituent by determining its speed and period and fixing the times of the maxima and minima of the corresponding tidal force. It usually consists of two parts represented by the symbols V and u. When referring to a particular instant of time such as the beginning of a series of observations, the V is written with a subscript as V_0. The first part of the argument includes any constant and multiples of one or more of the following astronomical elements—T, the hour angle of the mean sun at the place of observation; s, the mean longitude of the moon; h the mean longitude of the sun; and p, the longitude of the lunar perigee. The second part u includes multiples of one or both of the elements ξ and ν, which are functions of the longitude of the moon's node and vary slowly between small positive and negative limits throughout a 19-year cycle. In a series of observations covering a year or less they are treated as constants with values pertaining to the middle of the series. They do not affect the average speed or period of the constituent. Their values corresponding to each degree of N, the longitude of the moon's node, are included in table 6, formulas for their computation being given on p. 156.

72. The hourly speed of a constituent may be obtained by adding the hourly speeds of the elements included in the V of the argument. These elementary speeds will be found in table 1. The period of a constituent is obtained by dividing 360° by its speed. The approximate period is determined by the element of greatest speed contained in the argument. Thus, the hour angle T has a speed of 15° per mean solar hour and all constituents with a single T in their arguments have periods approximating one day, while constituents with arguments containing the multiple $2T$ have periods approximating the half day. Next to T, the element of greatest speed is s the mean longitude of the moon, and long-period constituents with a single s in their arguments will have periods approximating the month and with any multiple of s the corresponding fraction of a month. The arguments and speeds of the constituents are listed in table 2. Numerical values of the arguments for the beginning of each calendar year from 1850 to 2000 are given in table 15 for constituents used in the Coast and Geodetic Survey tide-predicting machine. Tables 16 to 18 provide differences for referring these arguments to any day and hour of the year.

73. In order to visualize the arguments of the constituents depending primarily upon the rotation of the earth, some have found it convenient to conceive of a system of fictitious stars, or "astres fictifs" as they are sometimes called, which move at a uniform rate in the celestial equator, each constituent being represented by a separate star. Thus, for the principal lunar constituent we have the mean moon and for the principal solar constituent the mean sun, while the various inequalities in the motions of these bodies are served by imaginary stars which reach the meridian of the place of observation at times corresponding to the zero value of the constituent argument. For the diurnal constituents the argument equals the hour angle of the star but for the semidiurnal constituents the argument is double the hour angle of the star.

74. *Coefficients.*—The complete coefficient of each term of formulas (62) to (64) includes several important factors. First, the *basic factor* U, which equals the ratio of the mass of the moon to that of the earth multiplied by the cube of the mean parallax of the moon, is common to all of the terms. This together with the common numerical coefficient may be designated as the *general coefficient*. Next, the function involving the latitude Y is known as the *latitude factor*, each formula having a different latitude factor. Following the latitude factor is a function of I, the inclination of the moon's orbit to the plane of the earth's equator, which may appropriately be called the *obliquity factor*, each factor applying to a group of terms. Lastly, we have an individual term coefficient which includes a numerical factor and involves the quantity e or m. Since these factors are derived from the equations of elliptic motion, they will here be referred to as *elliptic factors*. The product of the elliptic factor by the mean value of the obliquity factor is known as the *mean constituent coefficient* (C). Numerical values for these coefficients are given in table 2. Since all terms in any one of the formulas have the same general coefficient and latitude factor, their relative magnitudes will be proportional to their constituent coefficients. Terms of different formulas, however, have different latitude factors and their constituent coefficients are not directly comparable without taking into account the latitude of the place of observation.

75. The obliquity factors are subject to variations throughout an 18.6-year cycle because of the revolution of the moon's node. During this period the value of I varies between the limits of $\omega-i$ and $\omega+i$, or from 18.3° to 28.6° approximately, and the functions of I change accordingly. In order that tidal data pertaining to different years may be made comparable, it is necessary to adopt certain standard mean values for the obliquity factors to which results for different years may be reduced. While there are several systems of means which would serve equally well as standard values, the system adopted by Darwin in the early development of the harmonic analysis of tides has the sanction of long usage and is therefore followed. By the Darwin method, the mean for the obliquity factor is obtained from the product of the obliquity factor and the cosine of the elements ξ and ν appearing in the argument. This may be expressed as the mean value of the product $J \cos u$, in which J is the function of I in the coefficient and u the function of ξ and ν in the argument. Since u is relatively small and its cosine differs little from unity, the resulting mean will not differ greatly from the mean of J alone or from the function of I when given its mean value.

76. Using Darwin's system as described in section 6 of his paper on the Harmonic Analysis of Tidal Observations published in volume I of his collection of Scientific Papers (also in Report of the British Association for the Advancement of Science in 1883), the following mean values are obtained for the obliquity factors in formulas (62) to (64). These values were used in the computation of the corresponding constituent coefficients in table 2. The subscript $_0$ is here used to indicate the mean value of the function.

For terms A_1 to A_5 in formula (62)

$$[2/3-\sin^2 I]_0=(2/3-\sin^2 \omega)(1-3/2 \sin^2 i)=0.5021 \qquad (65)$$

For terms A_6 to A_{13} in formula (62)

$$[\sin^2 I \cos 2\xi]_0=\sin^2 \omega \cos^4 \tfrac{1}{2}i=0.1578 \qquad (66)$$

For terms A_{14} to A_{21} in formula (63)

$$[\sin I \cos^2 \tfrac{1}{2}I \cos (2\xi-\nu)]_0=\sin \omega \cos^2 \tfrac{1}{2}\omega \cos^4 \tfrac{1}{2}i=0.3800 \tag{67}$$

For terms A_{22} to A_{30} in formula (63)

$$[\sin 2I \cos \nu]_0=\sin 2\omega\ (1-3/2 \sin^2 i)=0.7214 \tag{68}$$

For terms A_{31} to A_{38} in formula (63)

$$[\sin I \sin^2 \tfrac{1}{2}I \cos (2\xi+\nu)]_0=\sin \omega \sin^2 \tfrac{1}{2}\omega \cos^4 \tfrac{1}{2}i=0.0164 \tag{69}$$

For terms A_{39} to A_{46} in formula (64)

$$[\cos^4 \tfrac{1}{2}I \cos (2\xi-2\nu)]_0=\cos^4 \tfrac{1}{2}\omega \cos^4 \tfrac{1}{2}i=0.9154 \tag{70}$$

For terms A_{47} to A_{55} in formula (64)

$$[\sin^2 I \cos 2\nu]_0=\sin^2 \omega\ (1-3/2 \sin^2 i)=0.1565 \tag{71}$$

For terms A_{56} to A_{63} in formula (64)

$$[\sin^4 \tfrac{1}{2}I \cos (2\xi+2\nu)]_0=\sin^4 \tfrac{1}{2}\omega \cos^4 \tfrac{1}{2}i=0.0017 \tag{72}$$

77. The ratio obtained by dividing the true obliquity factor for any value of I by its mean value may be called a *node factor* since it is a function of the longitude of the moon's node. The symbol generally used for the node factor is the small f. The node factor may be used with a mean constituent coefficient to obtain the true coefficient corresponding to a given longitude of the moon's node. Node factors for the several terms of formulas (62) to (64) may be expressed by the following ratios:

$$f(A_1) \text{ to } f(A_5)=f(\text{Mm})=(2/3-\sin^2 I)/0.5021 \tag{73}$$

$$f(A_6) \text{ to } f(A_{13})=f(\text{Mf})=\sin^2 I\ /0.1578 \tag{74}$$

$$f(A_{14}) \text{ to } f(A_{21})=f(\text{O}_1)=\sin I \cos^2 \tfrac{1}{2}I\ /0.3800 \tag{75}$$

$$f(A_{22}) \text{ to } f(A_{30})=f(\text{J}_1)=\sin 2I\ /0.7214 \tag{76}$$

$$f(A_{31}) \text{ to } f(A_{38})=f(\text{OO}_1)=\sin I \sin^2 \tfrac{1}{2}I\ /0.0164 \tag{77}$$

$$f(A_{39}) \text{ to } f(A_{46})=f(\text{M}_2)=\cos^4 \tfrac{1}{2}I\ /0.9154 \tag{78}$$

$$f(A_{47}) \text{ to } f(A_{55})=\sin^2 I\ /0.1565 \tag{79}$$

$$f(A_{56}) \text{ to } f(A_{63})=\sin^4 \tfrac{1}{2}\ I\ /0.0017 \tag{80}$$

Node factors for the middle of each calendar year from 1850 to 1999 are given in table 14 for the constituents used in the Coast and Geodetic Survey tide-predicting machine. These include all the factors above excepting formulas (79) and (80). However, since formula (79) represents an increase of only about one per cent over formula (74), the tabular values for the latter are readily adapted to formula (79). Node factors change slowly and interpolations can be made in table 14 for any desired part of the year. For practical purposes, however, the values for the middle of the year are generally taken as constant for the entire year.

78. The reciprocal of the node factor is called the *reduction factor* and is usually represented by the capital F. Applied to tidal coefficients pertaining to any particular year, the reduction factors serve to reduce them to a uniform standard in order that they may be comparable. Logarithms of the reduction factors for every tenth of a degree of I are given in table 12 for the constituents used on the tide-predicting machine of this office.

79. Formulas (62), (63), and (64), for the long-period, diurnal, and semidiurnal constituents of the vertical component of the tide-producing force may now be summarized as follows:

Let E=constituent argument from table 2
C=mean constituent coefficient from table 2
f=node factor from table 14

Then

$$F_{v30}/g=3/2\ U(1/2-3/2\ \sin^2 Y)\ \Sigma fC \cos E \qquad (81)$$

$$F_{v31}/g=3/2\ U \sin 2Y\ \Sigma fC \cos E \qquad (82)$$

$$F_{v32}/g=3/2\ U \cos^2 Y\ \Sigma fC \cos E \qquad (83)$$

Latitude factors for each degree of Y are given in table 3. The column symbol in this table is Y with annexed letter and digits corresponding to those in the designation of the tidal forces. Thus, Y_{v30} represents the latitude factor to be used with force F_{v30}, its value being equal to the function $(1/2-3/2\ \sin^2 Y)$. Taking the numerical value for the basic factor U from table 1, the general coefficient $3/2\ U$ is found to be 0.8373×10^{-7}.

HORIZONTAL COMPONENTS OF FORCE

80. The horizontal component of the principal part of the tide-producing force as expressed by formula (25), page 14, is in the direction of the azimuth of the tide-producing body. This component may be further resolved into a north-and-south and an east-and-west direction. In the following discussion the south and west will be considered as the positive directions for these components. Now let

F_{s3}/g = south component of principal tide-producing force
F_{w3}/g = west component of principal tide-producing force
A = azimuth of moon reckoned from the south through the west.

From formula (25), we then have

$$F_{s3}/g=3/2\ (M/E)(a/d)^3 \sin 2z \cos A \qquad (84)$$

$$F_{w3}/g=3/2\ (M/E)(a/d)^3 \sin 2z \sin A \qquad (85)$$

81. Referring to figure 3, page 16, the angle $P'PM$ equals A, the azimuth of the moon. Now, keeping in mind that the angle MPC is the supplement of A, the angle PCM equals t, and the arcs MC and PC are the respective complements of D and Y, we may obtain from the spherical triangle MPC the following relations:

$$\sin z \cos A=-\cos Y \sin D+\sin Y \cos D \cos t \qquad (86)$$

$$\sin z \sin A=\cos D \sin t \qquad (87)$$

Multiplying each of the above equations by the value of $\cos z$ from formula (31), the following equations may be derived:

$$\begin{aligned}\sin 2z \cos A&=2 \sin z \cos z \cos A\\ &=3/4 \sin 2Y\ (2/3-2\sin^2 D)\\ &\quad-\cos 2Y \sin 2D \cos t\\ &\quad+1/2 \sin 2Y \cos^2 D \cos 2t\end{aligned} \qquad (88)$$

$$\begin{aligned}\sin 2z \sin A&=2 \sin z \cos z \sin A\\ &=\sin Y \sin 2D \sin t\\ &\quad+\cos Y \cos^2 D \sin 2t\end{aligned} \qquad (89)$$

82. Substituting in (84) and (85) the quantities from equations (88) and (89), we have

$$\begin{aligned}F_{s3}/g&=9/8\ (M/E)(a/d)^3 \sin 2Y\ (2/3-2\sin^2 D) \text{ .... } F_{s30}/g\\ &\quad-3/2\ (M/E)(a/d)^3 \cos 2Y \sin 2D \cos t \text{ .... } F_{s31}/g\\ &\quad+3/4\ (M/E)(a/d)^3 \sin 2Y \cos^2 D \cos 2t \text{ .... } F_{s32}/g\end{aligned} \qquad (90)$$

$$\begin{aligned}F_{w3}/g&=3/2\ (M/E)(a/d)^3 \sin Y \sin 2D \sin t \text{ .... } F_{w31}/g\\ &\quad+3/2\ (M/E)(a/d)^3 \cos Y \cos^2 D \sin 2t \text{ .... } F_{w32}/g\end{aligned} \qquad (91)$$

The south component is expressed by three terms representing respectively the long-period, diurnal, and semidiurnal constituents. For the west component there are only two terms—the diurnal and semidiurnal, there being no long-period constituents in the west component. Each term has been marked separately by a symbol with annexed digits analogous to those used for the vertical component to indicate the class to which the term belongs.

83. Comparing formula (90) for the south component with formula (32) for the vertical component, it will be noted that the same functions of D and t are involved in the corresponding terms of both formulas, and that the terms differ only in their numerical coefficient and the latitude factor. Allowing for these differences, summarized formulas analogous to those given for the vertical component (page 26) may be readily formed. In order to eliminate the negative sign of the coefficient of the middle term, 180° will be applied to the arguments of that term. With all symbols as before, we then have

$$F_{s30}/g = 9/8\ U \sin 2Y\, \Sigma\, fC \cos E \tag{92}$$

$$F_{s31}/g = 3/2\ U \cos 2Y\, \Sigma\, fC \cos (E+180°) \tag{93}$$

$$F_{s32}/g = 3/4\ U \sin 2Y\, \Sigma\, fC \cos E \tag{94}$$

84. Comparing the two terms in formula (91) for the west component with the corresponding terms in formula (32) for the vertical component, it will be noted that the D functions are the same but that in (91) the sine replaces the cosine for the functions of t. It may be shown that the corresponding development of these terms will be the same as for the vertical component except that in the developed series each argument will be represented by its sine instead of cosine. In order that the summarized formulas may be expressed in cosine functions, 90° will be subtracted from each argument. With the same symbols as before and allowing for differences in the latitude factors, we obtain

$$F_{w31}/g = 3/2\ U \sin Y\, \Sigma\, fC \cos (E-90°) \tag{95}$$

$$F_{w32}/g = 3/2\ U \cos Y\, \Sigma\, fC \cos (E-90°) \tag{96}$$

85. Formulas for the horizontal component of tide-producing force in any given direction may be derived as follows: Let A equal the azimuth (measured from south through west) of given direction, and let F_{a30}/g, F_{a31}/g, and F_{a32}/g, respectively, represent the long-period, diurnal, and semidiurnal terms of the component in this direction. Then

$$F_{a30}/g = F_{s30}/g \times \cos A \tag{97}$$

$$F_{a31}/g = F_{s31}/g \times \cos A + F_{w31}/g \times \sin A \tag{98}$$

$$F_{a32}/g = F_{s32}/g \times \cos A + F_{w32}/g \times \sin A \tag{99}$$

As the long-period term has no west component, the summarized formula for the azimuth A may be derived by simply introducing the factor $\cos A$ into the coefficient of formula (92). For the diurnal and semidiurnal terms it is necessary to combine the resolved elements from the south and west components.

86. Referring to formulas (93) to (96) and considering a single constituent in each species we obtain the following:

Diurnal constituent,

$$3/2\ UfC\ [\cos 2Y \cos A \cos (E+180°)+\sin Y \sin A \cos (E-90°)]$$
$$=3/2\ UfC\ (-\cos 2Y \cos A \cos E+\sin Y \sin A \sin E)$$
$$=3/2\ UfC\ P_1 \cos (E-X_1) \quad (100)$$

in which

$$P_1=(\cos^2 2Y \cos^2 A+\sin^2 Y \sin^2 A)^{\frac{1}{2}} \quad (101)$$

$$X_1=\tan^{-1}\frac{\sin Y \sin A}{-\cos 2Y \cos A} \quad (102)$$

Semidiurnal constituent,

$$3/2\ UfC\ [\sin Y \cos Y \cos A \cos E+\cos Y \sin A \cos (E-90°)]$$
$$=3/2\ UfC\ \cos Y\ (\sin Y \cos A \cos E+\sin A \sin E)$$
$$=3/2\ UfC\ P_2 \cos (E-X_2) \quad (103)$$

in which

$$P_2=\cos Y\ (\sin^2 Y \cos^2 A+\sin^2 A)^{\frac{1}{2}} \quad (104)$$

$$X_2=\tan^{-1}\frac{\sin A}{\sin Y \cos A} \quad (105)$$

87. Summarized formulas for the horizontal component of the tide-producing force in any direction A may now be written as follows:

$$F_{a30}\ /g=9/8\ U \sin 2Y \cos A\ \Sigma fC \cos E \quad (106)$$

$$F_{a31}\ /g=3/2\ UP_1\ \Sigma fC \cos (E-X_1) \quad (107)$$

$$F_{a32}\ /g=3/2\ UP_2\ \Sigma fC \cos (E-X_2) \quad (108)$$

the values for P_1, P_2, X_1 and X_2 being obtained by formulas in the preceding paragraph. P_1 and P_2 are to be taken as positive and the following table will be found convenient in determining the proper quadrant for X_1 and X_2.

A quadrant	North latitude		South latitude	
	X1 quadrant	X2 quadrant	X1 quadrant	X2 quadrant
1	2 or 1	1	3 or 4	2
2	1 or 2	2	4 or 3	1
3	4 or 3	3	1 or 2	4
4	3 or 4	4	2 or 1	3

For the X_1 quadrant the first value of each pair is applicable when the latitude does not exceed 45° north or south. Otherwise the second value is applicable.

EQUILIBRIUM TIDE

88. The *equilibrium theory* of the tides is a hypothesis under which it is assumed that the waters covering the face of the earth instantly respond to the tide-producing forces of the moon and the sun and form a surface of equilibrium under the action of these forces. The theory disregards friction and inertia and the irregular distribution of the land masses of the earth. Although the actual tidal movement

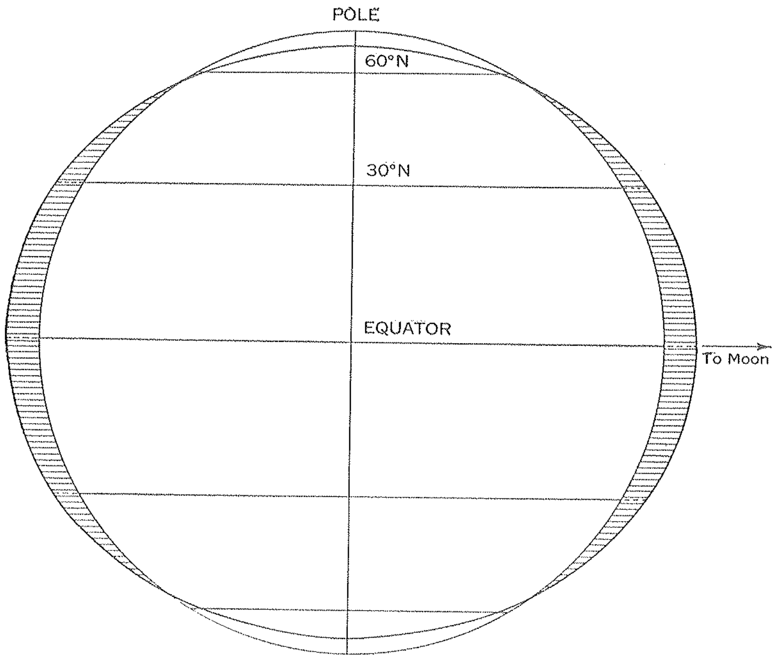

FIGURE 5.

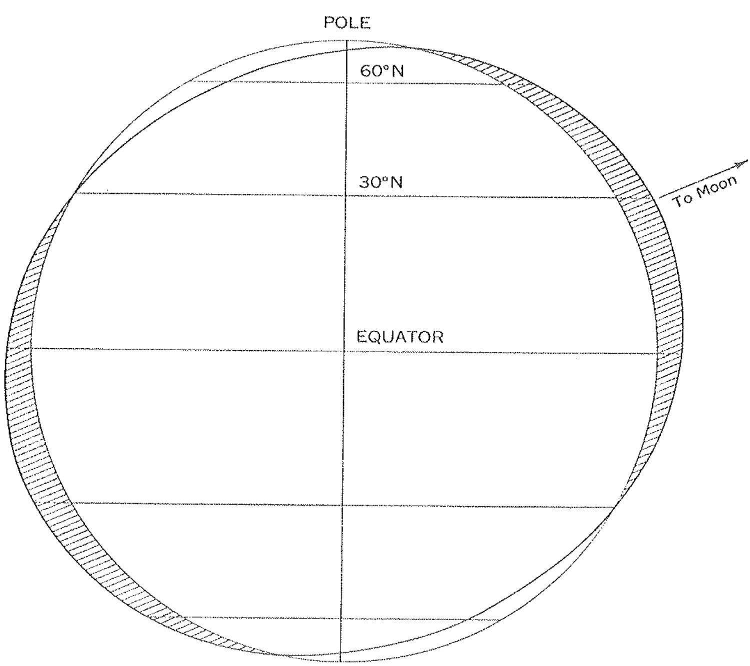

FIGURE 6.

of nature does not even approximate to that which might be expected under the assumed conditions, the theory is of value as an aid in visualizing the distribution of the tidal forces over the surface of the earth. The theoretical tide formed under these conditions is known as the *equilibrium tide*, and sometimes as the *astronomical* or *gravitational* tide.

89. Under the equilibrium theory, the moon would tend to draw the earth into the shape of a prolate spheroid with the longest axis in line with the moon, thus producing one high water directly under the moon and another one on the opposite side of the earth with a low water belt extending entirely around the earth in a great circle midway between the high water points. It may be shown mathematically, however, that the total effect of the moon at its mean distance would be to raise the high water points about 14 inches above the mean surface of the earth and depress the low water belt about 7 inches below this surface, giving a maximum range of tide of about 21 inches. The corresponding range due to the sun is about 10 inches. Figures 5 and 6 illustrate on an exaggerated scale the theoretical disturbing effect of the moon on the earth. In the first figure the moon is assumed to be directly over the equator and in the last figure the moon is approximately at its greatest north declination.

90. With the moon over the equator (fig. 5), the range of the equilibrium tide will be at a maximum at the equator and diminish to zero at the poles and at any point there will be two high and low waters of equal range with each rotation of the earth. With the moon north or south of the equator (fig. 6), a declinational inequality is introduced and the two high and low waters of the day for any given latitude would no longer be equal except at the equator. This inequality would increase with the latitude and near the poles only one high and low water would occur with each rotation of the earth. Although latitude is an important factor in determining the range of the equilibrium tide, it is to be kept in mind that in the actual tide of nature the latitude of a place has no direct effect upon the rise and fall of the water.

91. A surface of equilibrium is a surface at every point of which the sum of the potentials of all the forces is a constant. On such a surface the resultant of all the forces at each point must be in the direction of the normal to the surface at that point. If the earth were a homogeneous mass with gravity as the only force acting, the surface of equilibrium would be that of a sphere. Each additional force will tend to disturb this spherical surface, and the total deformation will be represented by the sum of the disturbances of each of the forces acting separately. In the following investigation we need not be especially concerned with the more or less permanent deformation due to the centrifugal force of the earth's rotation, since we may assume that the disturbances of this spheriodal surface due to the tidal forces will not differ materially from the disturbances in a true spherical surface due to the same cause.

92. The potential at any point due to a force is the amount of work that would be required to move a unit of matter from that point, against the action of the force, to a position where the force is zero. This amount of work will be independent of the path along which the unit of matter is moved. If the force being considered is the gravity of the earth the potential at any point will be the amount of work required to move a unit mass against the force of gravity from the point to an infinite distance from the earth's center. For the tide-producing force, the potential at any point will be measured by the amount of work necessary to move the unit of mass to the earth's center where this force is zero.

93. Referring to formula (21) for the vertical component of the tide-producing force, if the unit g is replaced by the unit μ from equation (15), the formula may be written as follows:

$$F_v = \frac{3\mu M}{d^3}(\cos^2 z - 1/3)r + \frac{3\mu M}{2d^4}(5\cos^3 z - 3\cos z)r^2 \qquad (109)$$

94. Considering separately the tide-producing potential due to the two terms in the above formula, let the potential for the first term involving the cube of the moon's distance be represented by V_3 and the potential for the second term involving the 4th power of the moon's distance by V_4. In each case the work required to move a unit mass against the force through an infinitesimal distance $-dr$ toward the center of the earth is the product of the force by $-dr$, and the potential or total work required to move the particle to the center of the earth may be obtained by integrating between the limits r and zero. Thus

$$V_3 = -\frac{3\mu M}{d^3}(\cos^2 z - 1/3)\int_r^0 r\,dr$$

$$= \frac{3\mu M}{2d^3}(\cos^2 z - 1/3)r^2 \qquad (110)$$

$$V_4 = -\frac{3\mu M}{2d^4}(5\cos^3 z - 3\cos z)\int_r^0 r^2\,dr$$

$$= \frac{\mu M}{2d^4}(5\cos^3 z - 3\cos z)r^3 \qquad (111)$$

95. At any instant of time the tide-producing potential at different points on the earth's surface will depend upon the zenith distance (z) of the moon and may be either positive or negative. It will now be shown that the average tide-producing potential for all points on the earth's surface, assuming it to be a sphere, is zero. Assume a series of right conical surfaces with common apex at center of earth and axis coinciding with the line joining centers of earth and moon, the angle between the generating line and the axis being z. These conical surfaces separated by infinitesimal angle dz will cut the surface of the sphere into a series of equipotential rings, the surface area of any ring being equal to a $2\pi r^2 \sin z\,dz$. The average potential for the entire spherical surface may then be obtained by summing the products of the ring areas and corresponding potentials and dividing the sum by the total surface area of the sphere. Thus

$$\text{Average } V_3 = \frac{3\mu M r^2}{4d^3}\int_0^\pi (\cos^2 z - 1/3)\sin z\,dz$$

$$= \frac{3\mu M r^2}{4d^3}\Big[-1/3\cos^3 z + 1/3\cos z\Big]_0^\pi = 0 \qquad (112)$$

$$\text{Average } V_4 = \frac{\mu M r^3}{4d^4} \int_0^\pi (5 \cos^3 z - 3 \cos z) \sin z \, dz$$

$$= \frac{\mu M r^3}{4d^4} \Big[-5/4 \cos^4 z + 3/2 \cos^2 z \Big]_0^\pi = 0 \qquad (113)$$

96. Let V_g represent the potential due to gravity at any point on the earth's surface. Since the force of gravity at any point on or above the earth's surface equals $\mu E/r^2$, the corresponding potential becomes

$$V_g = \mu E \int_r^\infty \frac{dr}{r^2} = \frac{\mu E}{r} \qquad (114)$$

If the earth is assumed to be a sphere with radius a, the gravitational potential at each point will equal $\mu E/a$, which may be taken as the average gravitational potential over the surface of the earth.

97. For a surface of equilibrium under the combined action of gravity and that part of the tide-producing force involving the cube of the moon's distance the sum of the corresponding potentials must be a constant, and since the average tide-producing potential for the entire surface of the earth is zero (par. 95), the constant will be the average gravitational potential or $\mu E/a$. Then from (110) and (114) we have

$$V_3 + V_g = \frac{3\mu M}{2d^3} (\cos^2 z - 1/3) r^2 + \frac{\mu E}{r} = \frac{\mu E}{a} \qquad (115)$$

Transposing and omitting common factor μ, we may obtain

$$\frac{(r-a)a^2}{r^3} = 3/2 (M/E)(a/d)^3 (\cos^2 z - 1/3) \qquad (116)$$

Let

$$r = a + h \qquad (117)$$

so that h represents the height of the equilibrium surface as referred to the undisturbed spherical surface of an equivalent sphere. Then

$$\frac{(r-a)a^2}{r^3} = \frac{ha^2}{(a+h)^3} = h/a - 3(h/a)^2 + 6(h/a)^3 - \text{etc.} \qquad (118)$$

As fraction h/a is very small, its greatest value being less than 0.000001, the powers above the first may be neglected. Substituting in (116) and writing h with subscript $_3$ to identify it with the principal tide-producing force, we have

$$h_3 / a = 3/2 \ (M/E)(a/d)^3 (\cos^2 z - 1/3) \qquad (119)$$

98. Similarly, for a surface of equilibrium under the combined action of gravity and the part of the tide-producing force involving the 4th power of the moon's distance, we have from (111) and (114)

$$V_4 + V_g = \frac{\mu M}{2d^4} (5 \cos^3 z - 3 \cos z) r^3 + \frac{\mu E}{r} = \frac{\mu E}{a} \qquad (120)$$

$$\frac{(r-a)a^3}{r^4}=1/2\ (M/E)(a/d)^4(5\cos^3 z-3\cos z) \qquad (121)$$

Letting $r=a+h_4$ and expanding the first member of the above formula, it becomes equal to h_4/a after the rejection of the higher powers of this small fraction. The formula may then be written

$$h_4/a=1/2\ (M/E)(a/d)^4(5\cos^3 z-3\cos z) \qquad (122)$$

99. Formulas (119) and (122) involving the cube and 4th power of the moon's parallax, respectively, represent the equilibrium heights of the tide due to the corresponding forces, the heights being expressed in respect to the mean radius (a) of the earth as the unit. In deriving these formulas the centrifugal force of the earth's rotation was disregarded and the resulting heights represent the disturbances in a true spherical surface due to the action of the tide-producing force. It may be inferred that in a condition of equilibrium the tidal forces would produce like disturbances in the spheroidal surface of the earth and the h of the formulas may therefore be taken as being referred to the earth's surface as defined by the mean level of the sea.

100. The extreme limits of the equilibrium tide, applicable to the time when the tide-producing body is nearest the earth, may be obtained by substituting the proper numerical values in formulas (119) and (122). They are given below for both moon and sun.

From formula (119) involving the cube of parallax—

Greatest rise =1.46 feet for moon, or 0.57 foot for sun (123)

Lowest fall =0.73 foot for moon, or 0.28 foot for sun (124)

Extreme range=2.19 feet for moon, or 0.85 foot for sun. (125)

From formula (122) involving the 4th power of parallax—

Greatest rise =0.026 foot for moon, or 0.000025 foot for sun (126)

Lowest fall =0.026 foot for moon, or 0.000025 foot for sun (127)

Extreme range=0.052 foot for moon, or 0.00005 foot for sun. (128)

101. A comparison of formulas (23) and (119), the first expressing the relation of the vertical component of the principal tide-producing force to the acceleration of gravity (g) and the other the relation of the height of the corresponding equilibrium tide to the mean radius (a) of the earth, will show that they are identical with the single exception that the coefficient of the height formula is one-half that of the force formula. Therefore the development of the force formula into a series of harmonic constituents is immediately applicable in obtaining similar expressions for the equilibrium height of the tide. Using a notation for the height terms corresponding to that used for the force terms, let h_{30}/a, h_{31}/a, and h_{32}/a represent, respectively, the long-period, diurnal, and semidiurnal terms of the equilibrium tide involving the cube of the moon's parallax. Then referring to formulas (81) to (83) we may write

$$h_{30}/a=3/4\ U(1/2-3/2\sin^2 Y)\ \Sigma fC\cos E \qquad (129)$$

$$h_{31}/a=3/4\ U\sin 2Y\ \Sigma fC\cos E \qquad (130)$$

$$h_{32}/a=3/4\ U\cos^2 Y\ \Sigma fC\cos E \qquad (131)$$

the symbols having the same significance as in the preceding discussion of the tidal forces.

TERMS INVOLVING 4TH POWER OF MOON'S PARALLAX

102. Formulas (24) and (26) represent the vertical and horizontal components of the part of the tide-producing force involving the 4th power of the moon's parallax. This part of the force constitutes only about 2 percent of the total tide-producing force of the moon and for brevity will be called the *lesser* force to distinguish it from the principal or primary part involving the cube of the parallax. The vertical component F_{v4}/g has its maximum value when z equals zero and, if numerical values pertaining to the moon and sun when nearest the earth are substituted in formula (24), the extreme values for this component are found to be 0.37×10^{-8} for the moon and 0.35×10^{-11} for the sun. The horizontal component F_{a4}/g has its greatest value when z equals about 31.09° and the substitution of numerical values in formula (26) gives the extreme value of this component as 0.26×10^{-8} for the moon or 0.24×10^{-11} for the sun.

103. Substituting in (24) the value of cos z from (31), the vertical component of the lesser force is expanded into four terms as follows:

$$
\begin{aligned}
F_{v4}/g = & \; 15/4\,(M/E)(a/d)^4 \sin Y(\cos^2 Y-2/5)\sin D(5\cos^2 D-2) \text{ - } F_{v40}/g \\
& +45/8\,(M/E)(a/d)^4 \cos Y(\cos^2 Y-4/5)\cos D(5\cos^2 D-4)\cos t \quad F_{v41}/g \\
& +45/4\,(M/E)(a/d)^4 \sin Y\cos^2 Y \sin D\cos^2 D\cos 2t \text{ - - - - - - - } F_{v42}/g \\
& +15/8\,(M/E)(a/d)^4 \cos^3 Y\cos^3 D\cos 3t \text{ - - - - - - - - - - - - - - - - } F_{v43}/g
\end{aligned}
\tag{132}
$$

These four terms represent, respectively, long-period, diurnal, semidiurnal, and terdiurnal constituents, according to the multiple of the hour angle t involved in the term. Each term is followed by a symbol which is analogous to those used in the development of the principal force.

104. Each term in formula (132) may be further expanded by means of the relations given in formulas (39) and (42). Expressing these terms separately we have—

$$
\begin{aligned}
F_{v40}/g = & \; 15/4\,(M/E)(a/d)^4 \sin Y(\cos^2 Y-2/5)\times \\
& [3(\sin I-5/4\sin^3 I)\cos(j-90°) \\
& +5/4\sin^3 I\cos(3j-90°)]
\end{aligned}
\tag{133}
$$

$$
\begin{aligned}
F_{v41}/g = & \; 45/8\,(M/E)(a/d)^4 \cos Y(\cos^2 Y-4/5)\times \\
& [5/4\sin I\cos^2 \tfrac{1}{2}I\cos(X-3j) \\
& +(1-10\sin^2\tfrac{1}{2}I+15\sin^4\tfrac{1}{2}I)\cos^2\tfrac{1}{2}I\cos(X-j) \\
& +(1-10\cos^2\tfrac{1}{2}I+15\cos^4\tfrac{1}{2}I)\sin^2\tfrac{1}{2}I\cos(X+j) \\
& +5/4\sin^2 I\sin^2\tfrac{1}{2}I\cos(X+3j)]
\end{aligned}
\tag{134}
$$

$$
\begin{aligned}
F_{v42}/g = & \; 45/8\,(M/E)(a/d)^4 \sin Y\cos^2 Y\times \\
& [\sin I\cos^4\tfrac{1}{2}I\cos(2X-3j+90°) \\
& +3(\cos^2\tfrac{1}{2}I-2/3)\sin I\cos^2\tfrac{1}{2}I\cos(2X-j-90°) \\
& +3(\cos^2\tfrac{1}{2}I-1/3)\sin I\sin^2\tfrac{1}{2}I\cos(2X+j-90°) \\
& +\sin I\sin^4\tfrac{1}{2}I\cos(2X+3j-90°)]
\end{aligned}
\tag{135}
$$

$$
\begin{aligned}
F_{v43}/g = & \; 15/8\,(M/E)(a/d)^4 \cos^3 Y\times \\
& [\cos^6\tfrac{1}{2}I\cos(3X-3j) \\
& +3\cos^4\tfrac{1}{2}I\sin^2\tfrac{1}{2}I\cos(3X-j) \\
& +3\cos^2\tfrac{1}{2}I\sin^4\tfrac{1}{2}I\cos(3X+j) \\
& +\sin^6\tfrac{1}{2}I\cos(3X+3j)]
\end{aligned}
\tag{136}
$$

105. If the common factor $(a/d)^4$ in formulas (133) to (136) is replaced by its equivalent $(a/c)^4\times(c/d)^4$, these formulas may be developed into numerous constituent terms by a method similar to that already described in the development of the principal lunar force (paragraphs 59–69). In the following development constituents of very small magnitude are omitted. Those given are numbered consecutively with the constituent terms of the principal lunar force.

$$F_{v40}/g=15/4\ (M/E)(a/c)^4 \sin Y\ (\cos^2 Y-2/5)\ \times$$

(A_{64}) $[(\sin I-5/4\sin^3 I)\{3(1+2e^2)\cos(s-90°-\xi)$

(A_{65}) $+9e\cos(2s-p-90°-\xi)$

(A_{66}) $+3e\cos(p-90°-\xi)\}$

(A_{67}) $+\sin^3 I\{5/4(1-6e^2)\cos(3s-90°-3\xi)$

(A_{68}) $+25/4\ e\cos(4s-p-90°-3\xi)\}]$ (137)

$$F_{v41}/g=45/8\ (M/E)(a/c)^4\cos Y\ (\cos^2 Y-4/5)\times$$

(A_{69}) $[\sin^2 I\cos^2 \tfrac{1}{2}I\{5/4(1-6e^2)\cos(T-3s+h+3\xi-\nu)$

(A_{70}) $+25/4\ e\cos(T-4s+h+p+3\xi-\nu)\}$

$+(1-10\sin^2\tfrac{1}{2}I+15\sin^4\tfrac{1}{2}I)\cos^2\tfrac{1}{2}I$

(A_{71}) $\{(1+2e^2)\cos(T-s+h+\xi-\nu)$ ---------- [M_1]

(A_{72}) $+3e\cos(T-2s+h+p+\xi-\nu)$

(A_{73}) $+e\cos(T+h-p+\xi-\nu)\}$

$+(1-10\cos^2\tfrac{1}{2}I+15\cos^4\tfrac{1}{2}I)\sin^2\tfrac{1}{2}I$

(A_{74}) $\{(1+2e^2)\cos(T+s+h-\xi-\nu)$

(A_{75}) $+3e\cos(T+2s+h-p-\xi-\nu)\}]$ (138)

$$F_{v42}/g=45/8\ (M/E)(a/c)^4\sin Y\cos^2 Y\ \times$$

(A_{76}) $[\sin I\cos^4\tfrac{1}{2}I\{(1-6e^2)\cos(2T-3s+2h+90°+3\xi-2\nu)$

(A_{77}) $+5e\cos(2T-4s+2h+p+90°+3\xi-2\nu)$

(A_{78}) $+e\cos(2T-2s+2h-p-90°+3\xi-2\nu)\}$

$+(\cos^2\tfrac{1}{2}I-2/3)\sin I\cos^2\tfrac{1}{2}I$

(A_{79}) $\{3(1+2e^2)\cos(2T-s+2h-90°+\xi-2\nu)$

(A_{80}) $+9e\cos(2T-2s+2h+p-90°+\xi-2\nu)\}$

$+(\cos^2\tfrac{1}{2}I-1/3)\sin I\sin^2\tfrac{1}{2}I$

(A_{81}) $\{3(1+2e^2)\cos(2T+s+2h-90°-\xi-2\nu)\}]$ (139)

$$F_{v43}/g=15/8\ (M/E)(a/c)^4\cos^3 Y\times$$

(A_{82}) $[\cos^6\tfrac{1}{2}I\{(1-6e^2)\cos(3T-3s+3h+3\xi-3\nu)$ ----- M_3

(A_{83}) $+5e\cos(3T-4s+3h+p+3\xi-3\nu)$

(A_{84}) $+e\cos(3T-2s+3h-p+180°+3\xi-3\nu)$

(A_{85}) $+127/8\ e^2\cos(3T-5s+3h+2p+3\xi-3\nu)$

(A_{86}) $+75/8\ me\cos(3T-4s+5h-p+3\xi-3\nu)\}$

$+\cos^4\tfrac{1}{2}I\sin^2\tfrac{1}{2}I$

(A_{87}) $\{3(1+2e^2)\cos(3T-s+3h+\xi-3\nu)$

(A_{88}) $+9e\cos(3T-2s+3h+p+\xi-3\nu)\}]$ (140)

106. All of the constituent terms in formulas (137) to (140) are relatively unimportant but they are listed in table 1 because of their theoretical interest. The only one of these terms now used in the prediction of tides is (A_{82}) representing the constituent M_3 which has a speed exactly three-halves that of the principal lunar constituent M_2. Term (A_{71}) is of interest in having a speed exactly one-half that of M_2 and is sometimes called the true M_1 to distinguish it from the composite M_1 which is used in the prediction of tides and which will be described later.

107. For simplicity and the purposes of this publication, the mean values of the obliquity factors in the terms of the lesser tide-producing force will be taken as the values pertaining to the time when I equals ω or 23.452°, excepting that for constituent M_3 and associated terms the mean has been obtained in accord with the system described in paragraph 75. The corresponding node factors (paragraph 77) may then be expressed by the following formulas in which the denominators are the accepted means of the obliquity factors:

$$f(A_{64}) \text{ to } f(A_{66}) = (\sin I - 5/4 \sin^3 I)/0.3192 \qquad (141)$$

$$f(A_{67}) \text{ to } f(A_{68}) = \sin^3 I/0.0630 \qquad (142)$$

$$f(A_{69}) \text{ to } f(A_{70}) = \sin^2 I \cos^2 \tfrac{1}{2} I/0.1518 \qquad (143)$$

$$f(A_{71}) \text{ to } f(A_{73}) = (1 - 10 \sin^2 \tfrac{1}{2} I + 15 \sin^4 \tfrac{1}{2} I) \cos^2 \tfrac{1}{2} I/0.5873 \qquad (144)$$

$$f(A_{74}) \text{ to } f(A_{75}) = (1 - 10 \cos^2 \tfrac{1}{2} I + 15 \cos^4 \tfrac{1}{2} I) \sin^2 \tfrac{1}{2} I/0.2147 \qquad (145)$$

$$f(A_{76}) \text{ to } f(A_{78}) = \sin I \cos^4 \tfrac{1}{2} I/0.3658 \qquad (146)$$

$$f(A_{79}) \text{ to } f(A_{80}) = (\cos^2 \tfrac{1}{2} I - 2/3) \sin I \cos^2 \tfrac{1}{2} I/0.1114 \qquad (147)$$

$$f(A_{81}) = (\cos^2 \tfrac{1}{2} I - 1/3) \sin I \sin^2 \tfrac{1}{2} I/0.0103 \qquad (148)$$

$$f(A_{82}) \text{ to } f(A_{86}) = f(M_3) = \cos^6 \tfrac{1}{2} I/0.8758 \qquad (149)$$

$$f(A_{87}) \text{ to } f(A_{88}) = \cos^4 \tfrac{1}{2} I \sin^2 \tfrac{1}{2} I/0.0380 \qquad (150)$$

Comparing formulas (149) and (78), it will be noted that the node factor for M_3 is equal to the node factor for M_2 raised to the 3/2 power. Computed values applicable to terms A_{82} to A_{86} are included in table 14 for years 1850 to 1999, inclusive.

108. For the tabulated constituent coefficients of the terms in formulas (137) to (140) there are included not only the elliptic and mean obliquity factors but also such other factors as may be necessary to permit the use of the general coefficient (3/2 U) of formulas (81) to (83) for the vertical component of the principal tide-producing force. The common coefficient $(M/E)\,(a/c)^4$ of formulas (137) to (140) is equal to U multiplied by the parallax a/c, and the latter together with the necessary numerical factors is included in the constituent coefficients in table 2. Formulas (137) to (140) may then be summarized as follows:

$$F_{v40}\ /g = 3/2\ U \sin Y(\cos^2 Y - 2/5)\ \Sigma fC \cos E \qquad (151)$$

$$F_{v41}\ /g = 3/2\ U \cos Y(\cos^2 Y - 4/5)\ \Sigma fC \cos E \qquad (152)$$

$$F_{v42}\ /g = 3/2\ U \sin Y \cos^2 Y\ \Sigma fC \cos E \qquad (153)$$

$$F_{v43}\ /g = 3/2\ U \cos^3 Y\ \Sigma fC \cos E \qquad (154)$$

109. It is to be noted that in formulas (151), (152), and (153), the maximum value of the latitude factor in each is less than unity, being

0.4, 0.2754, and 0.3849, respectively, if the sign of the function is disregarded. In formula (154), as in the corresponding formulas for the principal tide-producing force, the maximum value of this factor is unity. In comparing the relative importance of the various constituents of the tide-producing force the latitude factor should be included with the mean coefficient. Attention is also called to the fact that the relative importance of the constituents involving the 4th power of the moon's parallax is greater in respect to the vertical component of the tide-producing force than in respect to the height of the equilibrium tide. In table 2 the mean coefficients are taken comparable in respect to the vertical component of the tide-producing force and the constituent coefficients pertaining to the lesser force are therefore 50 percent greater than they would be if taken comparable in respect to the equilibrium tide.

110. The south and west horizontal components of the lesser tide-producing force may be obtained by multiplying formula (26) by cos A and sin A, respectively. Using the same system of notation as before, we then have

$$F_{s4}/g=3/2\ (M/E)(a/d)^4 \sin z\ (5 \cos^2 z-1) \cos A \tag{155}$$

$$F_{w4}/g=3/2\ (M/E)(a/d)^4 \sin z\ (5 \cos^2 z-1) \sin A \tag{156}$$

111. By means of the relations expressed in formulas (31), (86), and (87), the above component forces may be separated into long-period, diurnal, semidiurnal, and terdiurnal terms as follows:

South component,

$$F_{s40}/g=-15/4\ (M/E)(a/d)^4 \cos Y(\cos^2 Y-4/5) \sin D(5 \cos^2 D-2) \tag{157}$$

$$F_{s41}/g=45/8\ (M/E)(a/d)^4 \sin Y(\cos^2 Y-4/15) \cos D(5 \cos^2 D-4) \cos t \tag{158}$$

$$F_{s42}/g=-45/4\ (M/E)(a/d)^4 \cos Y(\cos^2 Y-2/3) \sin D \cos^2 D \cos 2t \tag{159}$$

$$F_{s43}/g=15/8\ (M/E)(a/d)^4 \sin Y \cos^2 Y \cos^3 D \cos 3t \tag{160}$$

West component,

$$F_{w41}/g=15/8\ (M/E)(a/d)^4(\cos^2 Y-4/5) \cos D(5 \cos^2 D-4) \sin t \tag{161}$$

$$F_{w42}/g=15/4\ (M/E)(a/d)^4 \sin 2Y \sin D \cos^2 D \sin 2t \tag{162}$$

$$F_{w43}/g=15/8\ (M/E)(a/d)^4 \cos^2 Y \cos^3 D \sin 3t \tag{163}$$

112. Comparing formulas (157) to (160) for the south component force with the corresponding terms of (132) for the vertical component, it will be noted that they differ only in the latitude factors and in sign for two of the terms. With adjustments for these differences the summarized formulas (151) to (154) are directly applicable for expressing the corresponding terms in the south component. Thus

$$F_{s40}/g=3/2\ U \cos Y(\cos^2 Y-4/5)\ \Sigma fC \cos(E+180°) \tag{164}$$

$$F_{s41}/g=3/2\ U \sin Y(\cos^2 Y-4/15)\ \Sigma fC \cos E \tag{165}$$

$$F_{s42}/g=3/2\ U \cos Y(\cos^2 Y-2/3)\ \Sigma fC \cos(E+180°) \tag{166}$$

$$F_{s43}/g=3/2\ U \sin Y \cos^2 Y\ \Sigma fC \cos E \tag{167}$$

113. For the west component there is no long-period term. Comparing (161) to (163) with the corresponding terms of (132), it will be noted that the t-functions are expressed as sines instead of cosines but they may be changed to the latter by subtracting 90° from each argument. With this change and allowing for differences in the latitude factors and numerical coefficients, the summarized formulas for the west component will be similar to those for the vertical component and may be written as follows:

$$F_{w41}/g = 1/2\ U\ (\cos^2 Y - 4/5)\ \Sigma fC \cos\ (E - 90°) \qquad (168)$$

$$F_{w42}/g = 1/2\ U \sin 2Y\ \Sigma fC \cos\ (E - 90°) \qquad (169)$$

$$F_{w43}/g = 3/2\ U \cos^2 Y\ \Sigma fC \cos\ (E - 90°) \qquad (170)$$

114. To obtain the horizontal component of the lesser force in any direction, the same procedure may be followed as was used for the principal tide-producing force (paragraphs 85 to 87). With the same system of notation we then have

$$F_{a40}/g = 3/2\ U \cos Y(\cos^2 Y - 4/5) \cos A\ \Sigma fC \cos\ (E + 180°) \qquad (171)$$

$$F_{a41}/g = 3/2\ U\ P_1\ \Sigma fC \cos\ (E - X_1) \qquad (172)$$

$$F_{a42}/g = 3/2\ U\ P_2\ \Sigma fC \cos\ (E - X_2) \qquad (173)$$

$$F_{a43}/g = 3/2\ U\ P_3\ \Sigma fC \cos\ (E - X_3) \qquad (174)$$

in which

$$P_1 = [\sin^2 Y(\cos^2 Y - 4/15)^2 \cos^2 A + 1/9(\cos^2 Y - 4/5)^2 \sin^2 A]^{1/2} \qquad (175)$$

$$P_2 = \cos Y[(\cos^2 Y - 2/3)^2 \cos^2 A + 4/9 \sin^2 Y \sin^2 A]^{1/2} \qquad (176)$$

$$P_3 = \cos^2 Y(\sin^2 Y \cos^2 A + \sin^2 A)^{1/2} \qquad (177)$$

$$X_1 = \tan^{-1} \frac{(\cos^2 Y - 4/5) \sin A}{3 \sin Y(\cos^2 Y - 4/15) \cos A} \qquad (178)$$

$$X_2 = \tan^{-1} \frac{2 \sin Y \sin A}{-3(\cos^2 Y - 2/3) \cos A} \qquad (179)$$

$$X_3 = \tan^{-1} \frac{\sin A}{\sin Y \cos A} \qquad (180)$$

The proper quadrants for X_1, X_2, and X_3 will be determined by the signs of the numerators and denominators in the above expressions, these signs being respectively the same as for the sine and cosine of the corresponding angles.

115. Comparing formula (122) for the equilibrium height of the tide due to the lesser tide-producing force with formula (24) for the vertical component of the force, it will be noted that they are the same with the exception that the numerical coefficient of the former is one-third that of the latter. With this change, the summarized formulas (151) to (154) for the vertical force may be used to express the corresponding equilibrium heights. Following the same system of notation as before, we have

$$h_{40}/a = 1/2\ U \sin Y(\cos^2 Y - 2/5)\ \Sigma fC \cos E \qquad (181)$$

$$h_{41}/a = 1/2\ U \cos Y(\cos^2 Y - 4/5)\ \Sigma fC \cos E \qquad (182)$$

$$h_{42}/a = 1/2\ U \sin Y \cos^2 Y\ \Sigma fC \cos E \qquad (183)$$

$$h_{43}/a = 1/2\ U \cos^3 Y\ \Sigma fC \cos E \qquad (184)$$

It is to be noted that the equilibrium height of the tide due to the principal tide-producing force when measured by the mean radius of the earth as a unit is one-half as great as the corresponding vertical component force referred to the mean acceleration of gravity as a unit, while the equilibrium height due to the lesser tide producing force similarly expressed is only one-third as great as the corresponding force. In table 2, the coefficients (C) of the constituents derived from the lesser force are made comparable with the others in respect to the vertical component force rather than in respect to the equilibrium height.

SOLAR TIDES

116. Since the tide-producing force of the sun is similar in action to that of the moon, the formulas derived for the latter are applicable, with suitable substitutions, to the solar forces. Referring to formulas (62), (63), and (64), let U be replaced by U_1 representing the product $(S/E)(a/c_1)^3$ in which S is the mass of the sun and (a/c_1) its mean parallax. Also replace e by e_1, the eccentricity of the earth's orbit; I by ω, the obliquity of the ecliptic; s by h, the mean longitude of the sun; and p by p_1, the longitude of the solar perigee. For the solar forces the arcs ξ and ν become zero and all terms representing the evectional and variational inequalities are omitted.

117. Making the changes indicated the solar constituents are now expressed in the following formulas. Each term is marked for identification by the letter B with the same subscript used for the corresponding term in the lunar tide. The usual constituent symbol is also given for the more important terms. Using the same system of notation as before,

Solar $F_{v30}/g = 3/2\ U_1\ (1/2-3/2\sin^2 Y)\times$

(B_1) $[(2/3-\sin^2\omega)\{(1+3/2\ e_1^2)$ .... permanent term
(B_2) $+3\ e_1\cos(h-p_1)$
(B_3) $+9/2\ e_1^2\cos(2h-2p_1)\}$
(B_6) $+\sin^2\omega\{(1-5/2\ e_1^2)\cos 2h$ .... Ssa
(B_7) $+7/2\ e_1\cos(3h-p_1)$
(B_8) $+1/2\ e_1\cos(h+p_1+180°)$
(B_9) $+17/2\ e_1^2\cos(4h-2p_1)\}]$ (185)

Solar $F_{v31}/g=3/2\ U_1\sin 2Y\times$

(B_{14}) $[\sin\omega\cos^2\frac{1}{2}\omega\{(1-5/2\ e_1^2)\cos(T-h+90°)$ ..... P_1
(B_{15}) $+7/2\ e_1\cos(T-2h+p_1+90°)$ .... π_1
(B_{16}) $+1/2\ e_1\cos(T-p_1-90°)$
(B_{17}) $+17/2\ e_1^2\cos(T-3h+2p_1+90°)\}$
(B_{22}) $+\sin 2\omega\{(1/2+3/4\ e_1^2)\cos(T+h-90°)$ .... $[K_1]$
(B_{23}) $+3/4\ e_1\cos(T+p_1-90°)$
(B_{24}) $+3/4\ e_1\cos(T+2h-p_1-90°)$ .... ψ_1
(B_{25}) $+9/8\ e_1^2\cos(T-h+2p_1-90°)$
(B_{26}) $+9/8\ e_1^2\cos(T+3h-2p_1-90°)\}$
(B_{31}) $+\sin\omega\sin^2\frac{1}{2}\omega\{(1-5/2\ e_1^2)\cos(T+3h-90°)$ ... ϕ_1
(B_{32}) $+7/2\ e_1\cos(T+4h-p_1-90°)$
(B_{33}) $+1/2\ e_1\cos(T+2h+p_1+90°)$
(B_{34}) $+17/2\ e_1^2\cos(T+5h-2p_1-90°)\}]$ (186)

Solar $F_{v32}/g=3/2\ U_1\cos^2 Y\times$

(B_{39}) $[\cos^4\frac{1}{2}\omega\{(1-5/2\ e_1^2)\cos(2T)$ .... S_2
(B_{40}) $+7/2\ e_1\cos(2T-h+p_1)$ .... T_2
(B_{41}) $+1/2\ e_1\cos(2T+h-p_1+180°)$ .... R_2
(B_{42}) $+17/2\ e_1^2\cos(2T-2h+2p_1)\}$
(B_{47}) $+\sin^2\omega\{(1/2+3/4\ e_1^2)\cos(2T+2h)$ .... $[K_2]$
(B_{48}) $+3/4\ e_1\cos(2T+h+p_1)$
(B_{49}) $+3/4\ e_1\cos(2T+3h-p_1)$

(Formula continued on next page)

(B_{50}) $\qquad +9/8\ e^2_1 \cos (2T+2p_1)$
(B_{51}) $\qquad +9/8\ e^2_1 \cos (2T+4h-2p_1)\}$
(B_{56}) $\qquad +\sin^4 \tfrac{1}{2}\omega\{(1-5/2\ e^2_1) \cos (2T+4h)$
(B_{57}) $\qquad +7/2\ e_1 \cos (2T+5h-p_1)$
(B_{58}) $\qquad +1/2\ e_1 \cos (2T+3h+p_1+180°)$
(B_{59}) $\qquad +17/2\ e^2_1 \cos (2T+6h-2p_1)\}]$ (187)

118. The general coefficient for the solar tide-producing force differs from that of the lunar force in the basic factor. From the fundamental data in table 1, the ratio of U_1/U is found to be 0.4602. This ratio, which will be designated as the *solar factor* with symbol S', represents the theoretical relation between the principal solar and lunar tide-producing forces. In computing the constituent coefficients of the solar terms for use in table 2, the solar factor was included in order that the same general coefficient may be applicable to both lunar and solar terms. All of the summarized formulas involving the coefficients and arguments of table 2 are therefore applicable to both lunar and solar constituents. For the solar constituents, however, the node factor (f) is always unity since ω, the obliquity of the ecliptic, may be considered as a constant.

119. By substituting solar elements in formulas (137) to (140) the corresponding solar constituents pertaining to the 4th power of the sun's parallax are readily obtained. Since the theoretical magnitude of the lesser solar tide-producing force is less than 0.00002 part of the total tide-producing force of moon and sun, it is usually disregarded altogether. However, certain interest is attached to three of the constituents which are considered in connection with shallow water and meteorological tides (p. 46). These are constituents Sa, S_1, and S_3, corresponding respectively to terms A_{64}, A_{71}, and A_{82} of the lunar series. They are listed in table 2 with reference letter B and corresponding subscripts. Sa has a speed one-half that of constituent Ssa represented by term B_6 of formula (185). Its theoretical argument as derived from term A_{64} contains the constant 90°, but being considered as a meteorological rather than an astronomical constituent, this constant is omitted from the argument. Constituents S_1 and S_3 have speeds respectively one-half and three-halves that of the principal solar constituent S_2.

120. The arguments of a number of the solar constituents include the element p_1 which represents the longitude of the solar perigee. As this changes less than 2° in a century, it may be considered as practically constant for the entire century. Referring to table 4 it will be noted that p_1 changes from 281.22° in 1900 to 282.94° in 2000. The value of 282° may therefore be adopted without material error for all work relating to the present century. With p_1 taken as a constant, it will be found that a number of terms in table 2 have the same speeds and may therefore be expected to merge into single constituents. Thus, constituents receiving contributions from more than one term are as follows: Sa from terms B_2, B_8, and B_{64}; Ssa from terms B_3 and B_6; P_1 from terms B_{14} and B_{25}; S_1 from terms B_{16}, B_{23}, and B_{71}; ψ_1 from terms B_{24} and B_{33}; ϕ_1 from terms B_{26} and B_{31}; S_2 from terms B_{39} and B_{50}; and R_2 from terms B_{41} and B_{48}. A few other solar terms also merge.

THE M_1 TIDE

121. The separation of constituents from each other by the process of the analysis depends upon the differences in their speeds. Constituents with nearly equal speeds are not readily separated unless the analysis covers a very long series of observations but they tend to merge and form a single composite constituent. In formula (63), terms A_{16} and A_{23} have nearly equal speeds, one being a little less and the other a little greater than one-half the speed of the principal lunar constituent M_2. These two terms are usually considered as a single constituent and represented by the symbol M_1. Neglecting for the present the general coefficient and common latitude factor, the two terms may be written as follows:

$$\text{term } A_{16}=1/2\ e \sin I \cos^2 \tfrac{1}{2}I \cos (T-s+h-p-90°+2\xi-\nu) \quad (188)$$

$$\text{term } A_{23}=3/2\ e \sin I \cos I \cos (T-s+h+p-90°-\nu) \quad (189)$$

The latter term, having a coefficient nearly three times as great as that of the first term, will predominate and determine the speed and period of the composite tide while the first term introduces certain inequalities in the coefficient and argument.

122. For brevity, let A and B represent the respective coefficients of terms A_{16} and A_{23} and let

$$\theta=T-s+h+p-90°-\nu \quad (190)$$

Also let P equal the mean longitude of the lunar perigee reckoned from the lunar intersection. Then

$$p=P+\xi \quad (191)$$

We then have

$$\begin{aligned}\text{term } A_{16}&=A \cos (\theta-2P)\\ &=A \cos 2P \cos \theta+A \sin 2P \sin \theta \end{aligned}\quad (192)$$

$$\text{term } A_{23}=B \cos \theta \quad (193)$$

$$\begin{aligned}M_1=A_{16}+A_{23}&=(A \cos 2P+B) \cos \theta+A \sin 2P \sin \theta\\ &=(A^2+2AB \cos 2P+B^2)^{\frac{1}{2}} \cos \left[\theta-\tan^{-1}\frac{A \sin 2P}{A \cos 2P+B}\right]\\ &=\frac{e \sin I \cos^2 \frac{1}{2}I}{Q_a} \cos (T-s+h+p-90°-\nu-Q_u)\end{aligned} \quad (194)$$

in which

$$1/Q_a=\left[1/4+3/2\frac{\cos I}{\cos^2 \frac{1}{2}I}\cos 2P+9/4\frac{\cos^2 I}{\cos^4 \frac{1}{2}I}\right]^{\frac{1}{2}} \quad (195)$$

$$Q_u=\tan^{-1}\frac{\sin 2P}{3 \cos I/\cos^2 \frac{1}{2}I+\cos 2P} \quad (196)$$

If I is given its mean value corresponding to ω, formula (195) may be reduced to the form

$$1/Q_a=(2.310+1.435 \cos 2P)^{\frac{1}{2}} \quad (197)$$

Values of log Q_a for each degree of P based upon formula (197) are given in table 9.

123. The period of the composite constituent M_1 is very nearly an exact multiple of the period of the principal lunar constituent M_2, and for this reason the summations which are necessary for the analysis of the latter may be conveniently adapted to the analysis of the former. With other symbols as before, let

$$\theta = T - s + h - 90° + \xi - \nu \tag{198}$$

Terms A_{16} and A_{23} may then be combined as follows:

$$\begin{aligned}\text{term } A_{16} &= A \cos(\theta - P) \\ &= A \cos P \cos \theta + A \sin P \sin \theta \end{aligned} \tag{199}$$

$$\begin{aligned}\text{term } A_{23} &= B \cos(\theta + P) \\ &= B \cos P \cos \theta - B \sin P \sin \theta \end{aligned} \tag{200}$$

$$\begin{aligned} M_1 = A_{16} + A_{23} &= (A+B) \cos P \cos \theta + (A-B) \sin P \sin \theta \\ &= (A^2 + 2AB \cos 2P + B^2)^{\frac{1}{2}} \cos\left[\theta - \tan^{-1}\left(\frac{A-B}{A+B} \tan P\right)\right] \\ &= \frac{e \sin I \cos^2 \frac{1}{2} I}{Q_a} \cos(T - s + h - 90° + \xi - \nu + Q) \end{aligned} \tag{201}$$

in which

$$Q = \tan^{-1}\left(\frac{5 \cos I - 1}{7 \cos I + 1} \tan P\right) \tag{202}$$

If I is given its mean value corresponding to ω, formula (202) may be reduced to the following form which was used for computing the values of Q in table 10.

$$\tan Q = 0.483 \tan P \tag{203}$$

124. Formulas (194) and (201) are the same except in the method of representing the argument. The elements $+p - Q_u$ in the first formula are replaced by $+\xi + Q$ in the latter, but it may be shown from (196) and (202) that

$$Q_u + Q = P = p - \xi \tag{204}$$

$$p - Q_u = \xi + Q \tag{205}$$

The complete arguments are therefore equal but in formula (201) the uniformly varying element p has been transferred from the V of the argument and included in the value of Q where it is treated as a constant for a series of observations being analyzed. The speed of the argument as determined by the remaining part of the V is then exactly one-half that of the principal constituent M_2 and with this assumption the summations for the latter may be adapted to the analysis of the former. It is to be noted, however, that the u in this case has a progressive forward change of nearly 41° each year. The true average speed of this constituent is determined by the V of formula (194) which includes the element p.

125. The obliquity factor for the composite M_1 constituent may be expressed by the formula $\sin I \cos^2 \frac{1}{2} I \times 1/Q_a$. According to the work of Darwin (Scientific Papers by Sir George H. Darwin, vol. 1, p. 39) the mean value of this factor is represented by the product $\sin \omega \cos^2 \frac{1}{2} \omega \cos^4 \frac{1}{2} i \times \sqrt{2.307}$, which equals 0.3800×1.52, or 0.5776. When deriving the node-factor formula for M_1, Darwin inadvertently omitted the factor $\sqrt{2.307}$ and obtained the approximate equivalent of the following:

$$f(M_1) = \frac{\sin I \cos^2 \frac{1}{2} I}{\sin \omega \cos^2 \frac{1}{2} \omega \cos^4 \frac{1}{2} i} \times 1/Q_a = \frac{\sin I \cos^2 \frac{1}{2} I}{0.3800} \times 1/Q_a \qquad (206)$$

Comparing the above with formula (75), it will be noted that

$$f(M_1) = f(O_1) \times 1/Q_a \qquad (207)$$

Factors pertaining to constituent M_1 in tables 13 and 14 are based upon the above formulas.

126. Because of the omission of the factor $\sqrt{2.307}$ from formula (206), the node factors for M_1 which have been in general use since this system of tidal reductions was adopted are about 50 percent greater than was originally intended, while the reciprocal reduction factors are correspondingly too small. This constituent is relatively unimportant and no practical difficulties have resulted from the omission. The M_1 amplitudes as reduced from the observational data are comparable among themselves but should be increased by 50 percent to be on the same basis as the amplitudes of other constituents. The predicted tides have not been affected in the least since the node factors and reduction factors are reciprocal and compensating. The theoretical mean coefficient for this constituent with the factor $\sqrt{2.307}$ included is 0.0317; but in order that this coefficient may be adapted for use with the tabular node factors when computing tidal forces or the equilibrium height of the tide, the coefficient 0.0209 with the factor $\sqrt{2.307}$ excluded should be used.

127. Although M_1 is one of the relatively unimportant constituents and the error in the node factor has caused no serious difficulties, it may be questionable whether it should be perpetuated. It is obvious, however, that any change in the present procedure would lead to much confusion unless undertaken by general agreement among all the principal organizations engaged in tidal work. By making any change applicable to the analysis of all series of observations beginning after a certain specified date it would be possible to interpret the results on the basis of the period covered by the observations without the necessity of revising all previously published amplitudes for this constituent.

THE L_2 TIDE

128. The composite L_2 constituent is formed by combining terms A_{41} and A_{48} of formula (64). Neglecting the general coefficient and common latitude factor these terms may be written

$$\text{term } A_{41} = 1/2\, e \cos^4 \tfrac{1}{2} I \cos (2T - s + 2h - p + 180° + 2\xi - 2\nu) \qquad (208)$$

$$\text{term } A_{48} = 3/4\, e \sin^2 I \cos (2T - s + 2h + p - 2\nu) \qquad (209)$$

A reference to table 2 will show that the mean coefficient of the first term is about four times as great as that of the latter term. The first term will therefore predominate and determine the speed of the composite constituent.

129. With other symbols as before, let A and B represent the respective coefficients of the two terms and θ the argument of the first term. We then have

$$A_{41}=A \cos \theta \tag{210}$$

$$A_{48}=B \cos (\theta+2P-180°)=-B \cos (\theta+2P) \tag{211}$$

$$L_2=A_{41}+A_{48}=(A-B \cos 2P) \cos \theta+B \sin 2P \sin \theta$$

$$=(A^2-2AB \cos 2P+B^2)^{\frac{1}{2}} \cos \left[\theta-\tan^{-1}\frac{B \sin 2P}{A-B \cos 2P}\right]$$

$$=1/2\ e\frac{\cos^4 \frac{1}{2}I}{R_a}\cos (2T-s+2h-p+180°+2\xi-2\nu-R) \tag{212}$$

in which

$$1/R_a=(1-12 \tan^2 \tfrac{1}{2}I \cos 2P+36 \tan^4 \tfrac{1}{2}I)^{\frac{1}{2}} \tag{213}$$

$$R=\tan^{-1} \frac{\sin 2P}{1/6 \cot^2 \frac{1}{2}I-\cos 2P} \tag{214}$$

Values of log R_a and R computed from the above formulas are given in tables 7 and 8, respectively.

130. The obliquity factor for the composite L_2 constituent may be expressed by the formula $\cos^4 \frac{1}{2}I\times 1/R_a$. The mean value of $1/R_a$ is approximately unity, and in accord with the Darwinian system the mean for the entire obliquity factor is taken as the product $\cos^4 \frac{1}{2}\omega \cos^4 \frac{1}{2}i$, which equals 0.9154 and is the same as the mean value of the obliquity factor for the principal constituent M_2. Multiplying this by the elliptic factor $\frac{1}{2}e$ gives 0.0251 as the mean constituent coefficient.

131. The node factor formula for constituent L_2 based upon the above mean for the obliquity factor is as follows:

$$f(L_2)=\frac{\cos^4 \frac{1}{2}I}{0.9145} \times 1/R_a=f(M_2) \times 1/R_a \tag{215}$$

Node factors for constituent L_2 based upon the above formula are included in table 14 for the middle of each year from 1850 to 1999, inclusive. The logarithms of the reciprocal reduction factors covering the period 1900 to 2000 are contained in table 13.

LUNISOLAR K_1 AND K_2 TIDES

132. Lunar diurnal term A_{22} of formula (63) and solar diurnal term B_{22} of formula (186) have the same speed. Together they form the lunisolar K_1 constituent. Also, lunar semidiurnal term A_{47} of formula (64) and solar semidiurnal term B_{47} of formula (187) have speeds exactly twice that of constituent K_1 and together form the lunisolar K_2 constituent. In order that the solar terms may have the same general coefficient as the lunar terms, the solar factor U_1/U, which will be designated by the symbol S', will be transferred from the general coefficient of the solar terms and included in the constituent coefficients. Then, neglecting the general coefficient and